12. UNSOLVED PROBLEMS.

We shall list here some problems which seem of interest to us, in the order of the chapters to which they refer. It appears that the most significant problem is problem 8. We know little about how the problems compare in difficulty, but some of the problems are only of technical interest.

To Chapter 3.

Problem 1. Prove that for bond-percolation on the triangular lattice with three parameters, as discussed in Application 3.4 (iii) the critical surface is

$$p(1) + p(2) + p(3) - p(1)\, p(2)\, p(3) = 1 \ . \qquad /// \tag{12.1}$$

Sykes and Essam (1964) conjectured that (12.1) gives the critical surface for this bond-percolation problem, and we mentioned several strong indications for the truth of this in Application 3.4 (iii). We also mentioned without proof that we can prove that for this problem

$$\theta(p) = 0 \ , \quad \text{whenever} \quad p >> 0 \quad \text{and} \quad p(1) + p(2) + p(3) - p(1)\, p(2)\, p(3) \leq 1. \tag{12.2}$$

The proof of this fact is based on the following theorem.

Theorem 12.1. Let $(\mathcal{G},\mathcal{G}^*)$ be a matching pair of periodic graphs in in $\mathbb{R}^2$ and let P_p be a λ-parameter periodic probability measure on the occupancy configurations of $\mathcal{G}$ based on the partition $\mathcal{V}_1,\dots\mathcal{V}_\lambda$ of the vertices of $\mathcal{G}$ (cf. Sect. 3.2). Assume that

$$P_p\{v \text{ is occupied}\} > 0 \quad \text{for all} \quad v.$$

Assume also that at least one of the following two symmetry conditions holds:

(i) the first or second coordinate axis is an axis of symmetry for $\mathcal{G}$ as well as for the partition $\mathcal{V}_1,\dots,\mathcal{V}_\lambda$ (cf. Def. 3.4),