LOCAL TRUTH

> *"a Grothendieck topology appears most naturally as a modal operator, of the nature 'it is locally the case that'"*
>
> F. W. Lawvere

The notion of a topological bundle represents but one side of the coin of sheaf theory. The other involves the conception of a sheaf as a functor defined on the category of open sets in a topological space. Our aim now is to trace the development of ideas that leads from this notion, via Grothendieck's generalisation, to the notion of a "topology" on a category and its attendant sheaf concept, and from there to the first-order concept of a topology on a topos and the resultant axiomatic sheaf theory of Lawvere and Tierney. The chapter is basically a survey, and its intention is to direct the reader to the appropriate literature.

14.1. Stacks and sheaves

Let I be a topological space, with Θ its set of open subsets. Θ becomes a poset category under the set inclusion ordering, in which the arrows are just the inclusions $U \hookrightarrow V$.

A *stack* or *pre-sheaf* over I is a *contravariant* functor from Θ to **Set**. Thus a stack F assigns to each open V a set $F(V)$, and to each inclusion $U \hookrightarrow V$ a function $F_U^V : F(V) \to F(U)$ (note the contravariance – reversal of arrow direction), such that

(i) $F_U^U = \mathrm{id}_U$, and
(ii) if $U \subseteq V \subseteq W$, then

$$\begin{tikzcd}
 & F(W) \arrow[dl, "F_V^W"'] \arrow[dr, "F_U^W"] & \\
F(V) \arrow[rr, "F_U^V"'] & & F(U)
\end{tikzcd}$$

commutes, i.e. $F_U^W = F_U^V \circ F_V^W$.