10. INEQUALITIES FOR CRITICAL PROBABILITIES.

We first give a theorem of Hammersley's (1961) stating that for any connected graph $\mathcal{G}$ the critical probability in a one-parameter problem for site-percolation on $\mathcal{G}(=p_H(\mathcal{G})$ in our notation) is at least as large as the critical probability for bond-percolation on $\mathcal{G}(=p_H(\tilde{\mathcal{G}})$, where $\tilde{\mathcal{G}}$ is the covering graph of $\mathcal{G}$; see Sect. 2.5). Actually, the result is obtained by comparing the probabilities that a fixed vertex z_0 is connected to some set of vertices V via a path with all vertices occupied, and via a path with all edges open, respectively. The proof given below is from Oxley and Welsh (1979). Hammersley (1980) has generalized this further to mixed bond and site problems (see Remark 10.1(i) below).

Special cases of the above mentioned inequality

$$p_H(\mathcal{G}) \geq p_H(\tilde{\mathcal{G}}) \tag{10.1}$$

are

(10.2) $p_H(\mathcal{G}_0)$ = critical probability for site-percolation on $\mathbb{Z}^2 \geq p_H(\mathcal{G}_1) = \frac{1}{2}$.

(see Ex. 2.1(i), 2.1(ii) and Application 3.4(ii)) and

(10.3) $p_H(\mathcal{T}) = \frac{1}{2} \geq$ critical probability for bond percolation on the triangular lattice $= 2\sin\frac{\pi}{18}$.

(see Ex. 2.1(iii) and Applications 3.4(i) and (iii)). In (10.3) we clearly have a strict inequality, and various data (Essam (1972)) indicate that $p_H(\mathcal{G}_0) \approx .59$ so that one long expected (10.2) to be a strict inequality as well. Higuchi (1982) recently gave the first proof of this strict inequality. Intuitively, the most important basis for a comparison of $p_H(\mathcal{G}_0)$ and $p_H(\mathcal{G}_1)$ is the fact that $\mathcal{G}_0$ can be