9. THE NATURE OF THE SINGULARITY AT p_H.

The arguments of Sykes and Essam (1964) which led them (not quite rigorously) to values for $p_H(\mathcal{G})$ for certain graphs were based on "the average number of clusters per site". In a one parameter problem with

(9.1) $$P_p\{v \text{ is occupied}\} = p$$

for all vertices v this average is, of course, a function, $\Delta(p,\mathcal{G})$ say, of p. Sykes and Essam's motivation for introducing this function lay in analogies with statistical mechanics, and on the basis of such analogies they assumed that $\Delta(p)$ has exactly one singularity as a function of p, and that this singularity is located at $p = p_H$. This assumption was actually their only non rigorous step. They then proved that for a matching pair of graphs $(\mathcal{G},\mathcal{G}^*)$ one has the remarkable relationship

(9.2) $$\Delta(p,\mathcal{G}) - \Delta(1-p,\mathcal{G}^*) = \text{a polynomial in } p.$$

It was for this relation that Sykes and Essam introduced matching pairs of graphs. They then proceeded to locate p_H, which was presumably the singularity of Δ, by means of (9.2) for certain matching pairs in which $\mathcal{G}$ and $\mathcal{G}^*$ have a close relation. E.g. for bond percolation on $\mathbb{Z}^2$, $\mathcal{G}_1^*$ is isomorphic to $\mathcal{G}_1$ and hence $\Delta(\cdot,\mathcal{G}_1) = \Delta(\cdot,\mathcal{G}_1^*)$.

In this chapter we shall first give the precise definition and show the existence of $\Delta(p)$, following Grimmett (1976) and Wierman (1978). We then derive the Sykes-Essam relation (9.2) and show that for the matching pairs $(\mathcal{G},\mathcal{G}^*)$ to which Theorem 3.1 applies $\Delta(p,\mathcal{G})$ is analytic in p for $p \neq p_H(\mathcal{G})$. This justifies part of the Sykes-Essam assumption: For various matching pairs $\Delta(\cdot,\mathcal{G})$ has at most one singularity, and if there is one it must be at $p_H(\mathcal{G})$. Unfortunately we have been unable to show that $\Delta(\cdot,\mathcal{G})$ has any singularity at p_H as a function of p only. (There is an obvious singularity if one brings in additional variables; compare the study of the function $f(h)$ in Kunz