7. PROOFS OF THEOREMS 3.1 AND 3.2.

The first step is to show that for a parameter point p_0 which satisfies Condition A or B of Ch. 3 there exist large rectangles for which the crossing probabilities in both the horizontal and vertical direction are bounded away from zero. The RSW theorem will then show that with P_{p_0}-probability one there exist arbitrarily large occupied circuits on $\mathcal{G}$ surrounding the origin. From this it follows that there are no infinite vacant clusters on $\mathcal{G}^*$ under P_{p_0}[1]. An interchange of the roles of $\mathcal{G}$ and $\mathcal{G}^*$ and of occupied and vacant then shows that there is also no percolation on $\mathcal{G}$ under P_{p_0}. This is just the content of (3.43), which is the most important statement in Theorem 3.1(i). Clearly the above implies that for p', p'' such that $p'(i) \le p_0(i) \le p''(i)$, $i = 1,\dots,\lambda$ also

$$P_{p'}\{\#W(v) = \infty\} = 0 \quad \text{and} \quad P_{p''}\{\#W^*(v) = \infty\} = 0.$$

The above conclusions are basically already in Harris' beautiful paper (Harris (1960)). The first proof that percolation actually occurs for $p'' >> p_0$ is in Kesten (1980a). The proof given below is somewhat simpler because we now use Russo's formula (Prop. 4.2) which only appeared in Russo (1981). Actually we prove the dual statement that for $p' << p_0$ infinite vacant clusters occur on $\mathcal{G}^*$. An easy argument shows that it suffices to show $E_{p'}\{\#W(v)\} < \infty$, and by Theorem 5.1 this will follow once we prove that the crossing-probabilities $\tau(\overline{N};1,p')$ and $\tau(\overline{N};2,p')$ of some large rectangles are small for $p' << p$. This is done by showing that $\frac{d}{dt}\tau(\overline{N};i,p(t))$ is "large" for $0 \le t \le 1$, $p(t) = (1-t)p' + tp_0$. By Russo's formula, this amounts to showing that there are many pivotal sites (see Def. 4.2) for the events

(7.1) $\quad A(\overline{N};i) = \{\ \exists$ occupied crossing in the i-direction of $T(\overline{N};i)\}$.

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1) In this part we shall use some simplifications suggested by S. Kotani.