6. THE RUSSO-SEYMOUR-WELSH THEOREM.

The object of this chapter is a result which states that if the crossing probabilities of certain rectangles in both the horizontal and vertical direction are bounded away from zero, then so are the crossing probabilities for larger rectangles. This result will then be used to prove the existence of occupied circuits surrounding the origin. The idea is to connect an occupied horizontal crossing of $[0,n_1] \times [0,n_2]$ and an occupied horizontal crossing of $[m,n_1 + m] \times [0,n_2]$ by means of a suitable occupied vertical crossing, in order to obtain a horizontal crossing of $[0,n_1 + m] \times [0,n_2]$. This would be quite simple (compare the proof of Lemma 6.2) if one had a lower bound for the probability of an occupied vertical crossing of $[m,n_1] \times [0,n_2]$, but in the applications one only has estimates for the existence of occupied vertical crossings of rectangles which are wider and/or lower. One therefore has to use some trickery, based on symmetry to obtain the desired connections. Such tricks were developed independently by Russo (1978) and Seymour and Welsh (1978). (See also Smythe and Wierman (1978), Ch. 3 and Russo (1981).) These papers dealt with the one-parameter problems on the graphs $\mathcal{G}_0$ or $\mathcal{G}_1$ (see Ex. 2.1(i) and (ii)) and therefore had at their disposal symmetry with respect to both coordinate axes, as well as invariance of the problem under interchange of the horizontal and vertical direction. We believe that neither of these properties is necessary, but so far we still need at least one axis of symmetry. We also have to restrict ourselves to a planar modification $\mathcal{G}_{p\ell}$ of a graph $\mathcal{G}$ which is one of a matching pair of graphs in $\mathbb{R}^2$.

Throughout this chapter we deal with the following setup:

(6.1) $(\mathcal{G},\mathcal{G}^*)$ is a matching pair based on $(\mathcal{M},\mathcal{F})$ for some mosaic $\mathcal{M}$ satisfying (2.1)-(2.5) and subset $\mathcal{F}$ of its collection of faces (see Sect. 2.2). $\mathcal{G}_{p\ell}$ is the planar modification of $\mathcal{G}$ (see Sect. 2.3).