CHAPTER 16

LOGICAL GEOMETRY

> *"It is a very interesting fact that notions originally developed for the purposes of (abstract) algebraic geometry turn out to be intimately related to logic and model theory. Compared to other existing versions of algebraic logic, categorical logic has the distinction of being concerned with objects that appear in mathematical practice."*
>
> Michael Makkai and Gonzalo Reyes

The theory discussed in this book emerges from an interaction between sheaf theory and logic, and for the most part we have dwelt on the impact of the former on the conceptual framework of the latter. In this chapter we will consider ways in which the application has gone in the opposite direction. Specifically, we study the concept of a *geometric morphism*, a certain kind of functor between topoi that plays a central role in the work of the Grothendieck school (Artin et al. [SGA 4]). In their book *First Order Categorical Logic*, henceforth referred to as [MR], Makkai and Reyes have shown that this notion of morphism can be reformulated in logical terms, and that some important theorems of Pierre Deligne and Michael Barr about the existence of geometric morphisms can be derived by model-theoretic constructions. The essence of their approach is to associate a *theory* (set of axioms) with a given site, and identify functors defined on the site with models of this theory. Conversely, from a certain type of theory a site can be built by a method that adds a new dimension of mathematical significance to the well-known Lindenbaum-algebra construction (cf. §6.5).

These developments will be described below, with our main aim being to account for the fact that Deligne's theorem is actually equivalent to a version of the classical Gödel Completeness Theorem for **Set**-based semantics of first-order logic.