CHAPTER 12

CATEGORIAL SET THEORY

> "... *the mathematics of the future, like that of the past, will include developments which are relevant to the philosophy of mathematics.... They may occur in the theory of categories where we see, once again, a largely successful attempt to reduce all of pure mathematics to a single discipline*".
>
> Abraham Robinson

While a topos is in general to be understood as a "generalised universe of sets", there are, as we have seen, many topoi whose structure is markedly different from that of **Set**, the domain of classical set theory. Even within a topos that has classical logic (is Boolean) there may be an infinity of truth-values, non-initial objects that lack elements, distinct arrows not distinguished by elements of their domain etc. So in order to identify those topoi that "look the same" as **Set** we will certainly impose conditions like well-pointedness and (hence) bivalence.

However, in order to say precisely which topoi look like **Set** we have to know precisely what **Set** looks like. Thus far we have talked blithely about *the* category of all sets without even acknowledging that there might be some doubt as to whether, or why, such a unique thing may exist at all. We resolve (sidestep?) this matter by introducing a formal first-order language for set-theory, in which we write down precise versions of set-theoretic principles. Instead of referring to "the universe **Set**", we confine ourselves to discussion of interpretations of this language. The notion of a topos is also amenable to a first-order description, as indicated in the last chapter, and so the relationship between topos theory and set theory can be rigorously analysed in terms of the relationship between models of two elementary theories.

Before looking at the details of this program we need to develop two more fundamental aspects of the category of sets.