FUNCTORS

> *"It should be observed first that the whole concept of a category is essentially an auxiliary one; our basic concepts are essentially those of a functor and of a natural transformation."*
>
> S. Eilenberg and S. MacLane

9.1. The concept of functor

A functor is a transformation from one category into another that "preserves" the categorial structure of its source. As the quotation from the founders of the subject indicates, the notion of functor is of the very essence of category theory. The original perspective has changed somewhat, and as far at least as this book is concerned functors are not more important than categories themselves. Indeed the viability of the topos concept as a foundation for mathematics pivots on the fact that it can be *defined* without reference to functors. However we have now reached the stage where we can ignore them no longer. They provide the necessary language for describing the relationship between topoi and Kripke models, and between topoi and models of set theory.

A *functor* F from category $\mathscr{C}$ to category $\mathscr{D}$ is a function that assigns

(i) to each $\mathscr{C}$-object a, a $\mathscr{D}$-object $F(a)$;

(ii) to each $\mathscr{C}$-arrow $f: a \to b$ a $\mathscr{D}$-arrow $F(f): F(a) \to F(b)$,

such that

(a) $F(1_a) = 1_{F(a)}$, all $\mathscr{C}$-objects a, i.e. the identity arrow on a is assigned the identity on $F(a)$,

(b) $F(g \circ f) = F(g) \circ F(f)$, whenever $g \circ f$ is defined.

This last condition states that the F-image of a composite of two arrows is the composite of their F-images, i.e. whenever