ALGEBRA OF SUBOBJECTS

"Since new paradigms are born from old ones, they ordinarily incorporate much of the vocabulary and apparatus, both conceptual and manipulative, that the traditional paradigm had previously employed. But they seldom employ these borrowed elements in quite the traditional way."

Thomas Kuhn

7.1. Complement, intersection, union

At the beginning of Chapter 6 it was asserted that the structure of $(\mathcal{P}(D), \subseteq)$ as **BA** depends on the rules of classical logic, through the properties of the connectives "and", "or", and "not". This can be made quite explicit by the consideration of characteristic functions. We see from the following result just how set operations depend on truth-functions.

THEOREM 1. *If A and B are subsets of D, with characters* $\chi_A : D \to 2$, $\chi_B : D \to 2$, *then*

(i) $\chi_{-A} = \neg \circ \chi_A$

(ii) $\chi_{A \cap B} = \chi_A \frown \chi_B \qquad (= \frown \circ \langle \chi_A, \chi_B \rangle)$

(iii) $\chi_{A \cup B} = \chi_A \smile \chi_B$.

PROOF. If $\chi_{-A}(x) = 1$, for $x \in D$, then $x \in -A$, so $x \notin A$, whence $\chi_A(x) = 0$, so $\neg\chi_A(x) = 1$. But if $\chi_{-A}(x) = 0$, then $x \notin -A$, so $x \in A$, whence $\chi_A(x) = 1$ and $\neg\chi_A(x) = 0$. Thus χ_{-A} and $\neg \circ \chi_A$ give the same output for the same input, and are identical. The proofs of (ii) and (iii) follow similar lines, using the definitions of $\cap$, $\frown$, $\cup$, $\smile$. □

Theorem 1 suggests a generalisation – the result in one context becomes the definition in another, as follows.