CHAPTER 6

LOGIC CLASSICALLY CONCEIVED

> *"It is not easy, and perhaps not even useful, to explain briefly what logic is."*
>
> E. J. Lemmon

6.1. Motivating topos logic

In any systematic development of set theory one of the first topics to be examined is the so-called *algebra of classes*. This is concerned with ways of defining new sets, and when relativised to the subsets of a given set D focuses on the operations of

Intersection: $A \cap B = \{x \colon x \in A \text{ and } x \in B\}$
Union: $A \cup B = \{x \colon x \in A \text{ or } x \in B\}$
Complement: $-A = \{x \colon x \in D \text{ and not } x \in A\}$

The power set $\mathcal{P}(D)$ together with the operations $\cap, \cup, -$ exhibit the structure of what is known as a *Boolean algebra*. These algebras, to be defined shortly, are intimately connected with the classical account of logical truth.

Now the operations $\cap, \cup, -$ can be characterised by universal properties, and hence defined in any topos, yielding an "algebra of subobjects". It turns out that in some cases, this algebra does not satisfy the laws of Boolean algebra, indicating that the "logic" of the topos is not the same as classical logic. The proper perspective, it would seem, is that the algebra of subobjects is non-Boolean *because* the topos logic is non-classical, rather than the other way round. In defining $\cap, \cup, -$ we used the words "and", "or", and "not", and so the properties of the set operations are determined by the meaning, the logical behaviour, of these words. It is the rules of classical logic that dictate that $\mathcal{P}(D)$ should be a Boolean algebra.

The classical rules of logic are representable in **Set** by operations on the set $2 = \{0, 1\}$, and can then be developed in any topos $\mathscr{E}$ by using Ω in place of 2. This gives the "logic" of $\mathscr{E}$, which proves to characterise the