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Computation of principal $\mathcal{A}$-determinants through dimer dynamics

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Abstract.

$\mathcal{A}$ is a set of N vectors in $\mathbb{Z}^{N-2}$ situated in a hyperplane not through 0 and spanning $\mathbb{Z}^{N-2}$ over $\mathbb{Z}$. Gulotta's algorithm [4] constructs from $\mathcal{A}$ a dimer model. A theorem in [6] states that the principal $\mathcal{A}$-determinant equals the determinant of (a suitable form of) the Kasteleyn matrix of that dimer model. In the present note we translate Gulotta's pictorial description of the algorithm into matrix operations. As a result one obtains an algorithm for computing the principal $\mathcal{A}$-determinant, which is much faster than the algorithm in [5].

§1. Introduction

$\mathcal{A} = \{\mathbf{a}_1, \dots, \mathbf{a}_N\}$ is a set of vectors in $\mathbb{Z}^{N-2}$ situated in a hyperplane not through 0 and spanning $\mathbb{Z}^{N-2}$ over $\mathbb{Z}$. The principal $\mathcal{A}$-determinant, defined in [2], describes the singularities of Gelfand–Kapranov–Zelevinsky's $\mathcal{A}$-hypergeometric system of partial differential equations [3]. It also describes for which N-tuples of coefficients $u_1, \dots, u_N \in \mathbb{C}$ the Laurent polynomial $\sum_{j=1}^{N} u_j \mathbf{x}^{\mathbf{a}_j}$ in $N-2$ variables is singular (see [2] for details). It is a polynomial with integer coefficients in the variables $u_1, \dots, u_N$. The restriction $\operatorname{rank} \mathcal{A} = \sharp\mathcal{A} - 2$ means that the corresponding hypergeometric functions are essentially functions in two variables and that in the Laurent polynomial the number of terms exceeds the number of variables by 2. Even in this case the definition of the principal $\mathcal{A}$-determinant is fairly complicated and only a few of its coefficients could explicitly be calculated in [2]. In [1] Dickenstein and Sturmfels re-examined the definition of the principal $\mathcal{A}$-determinant and related

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