Tangent Space, Metric, and Directional Derivative

In this chapter we will begin to set up the formal machinery that will allow us to talk about curvature on a surface. We will use this terminology and formalism to obtain an equation for the normal curvature of any curve on the surface. (See Problem **4.7**.) In Chapter 5 we will use this formalism as a part of our intrinsic description of intrinsic curvature. The expression in Problem **4.7** for the normal curvature at a point on a curve depends only on the direction of the curve at that point. Thus it will be the starting point for our investigation of the curvature of the surface in Chapter 6. Note from Chapter 3 we know that the normal curvature is due to the curving of the surface and not due to any intrinsic curving of curves in the surface.

PROBLEM *4.1. The Tangent Space*

Go back to Problem **3.1** for the discussion of smooth surfaces and their tangent spaces and normal spaces.

If a curve C intersects a plane Π at a point p, we say that C is **tangent** to the plane at p, if when we zoom in on p sufficiently [that is, given any tolerance τ there is radius ρ such that in any f.o.v. with radius $< \rho$] the portion of the curve in the f.o.v. is indistinguishable from a subset of the plane [and the projection of the curve to the plane is 1-1]. But clearly this does not mean that C lies in the plane. Thus, in general, for a curve that is tangent to the plane at p, as we zoom in, the portion of the curve in the f.o.v. becomes closer and closer to the plane until it becomes indistinguishable from it. However, when the curve is straight then, as we zoom in, we see the same picture at all magnifications. (See Figure 4.1.) Which angles we can distinguish depend on the tolerance. With decreasing tolerances we will be able to distinguish smaller and smaller angles. Put this discussion together to show that:
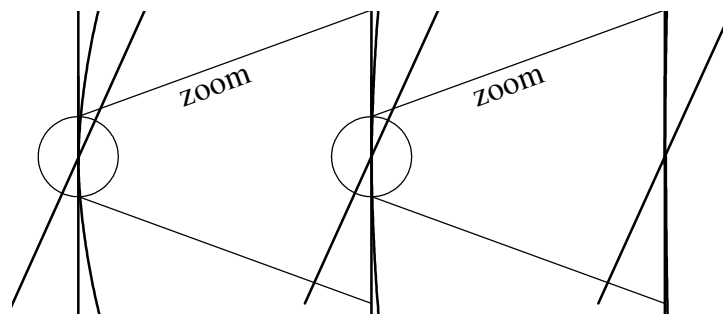

Figure 4.1. Zooming In on Two Straight Lines and a Tangent Curve

    a. *A straight line that is tangent to a plane is contained in the plane* (not merely indistinguishable from it).

Thus, note that tolerances must increase as we zoom in. This is not the same as "zooming in" in computer graphics because the normal computer screen has fixed pixels and thus fixed tolerance. To model on a