SOLUTIONS

Chapter 4

Tangent Space, Metric, and Directional Derivative

PROBLEM 4.1. The Tangent Space

a.

If we have a curve C which intersects a plane Π at a point p, then, if we zoom in on p and find that the portion of the curve in the f.o.v. becomes indistinguishable from a subset of the plane, then we would say that C is *tangent* to the plane at p. But clearly this does not mean that C lies in the plane. Thus, in general, for a curve that is tangent to the plane at p, as we zoom in, the portion of the curve in the f.o.v. becomes closer and closer to the plane until it becomes indistinguishable from it. However, when the curve is straight (such as a vector) then as we zoom in, we see the same picture at all magnifications. See Figure 4.1 of the text. Which angles we can distinguish depend on the tolerance. With decreasing tolerances we will be able to distinguish smaller and smaller angles. In fact the tolerance is essentially a measure of the smallest angle (subtended at the center) that is not indistinguishable from a line segment. Thus, straight line is tangent to the plane only if it forms a zero angle with the plane and, thus, is in the plane.

b.

The velocity vector will be tangent to the surface and, thus, to the tangent plane. By Part **a** this vector then must lie in the plane.

c.

Consider the intersection of M with $(n-1)$-dimensional subspaces determined by a tangent vector in $T_{\mathbf{p}}M$ and the whole normal space $N_{\mathbf{p}}M$. See Figure 4.2 (in the text) for a picture of this situation in $\mathbf{R}^3$. By the definition of smooth surface, near p the surface M projects one-to-one onto the tangent plane; thus, the intersection of the $(n-1)$-dimensional subspace with M is a curve (near p), which we call C.

Pick rectangular coordinates for $\mathbf{R}^n$ so that $p=(0,0,0,...,0)$ and $\mathbf{V}=(|\mathbf{V}|,0,0,...,0)$. Then let $g(a,b,c,...,z) = (a,0,0,...,0)$ be the projection onto the tangent line at p. Then $g|C$ (the projection restricted to C) is one-to-one and there is a function $\gamma\colon \mathbf{R} \to C$ such that $g(\gamma(t)) = (|\mathbf{V}|t,0,0,...,0)$. This γ gives a parametrization for a neighborhood of p in C and $\gamma'(0) = \mathbf{V}$.

PROBLEM 4.2. Mean Value Theorem - Curves - Surfaces

a.

Look at the line determined by **p** and **q** and then move this line parallel to itself (in one or the other direction) until it last touches the curve. Call this parallel transported line of last contact l. The point **r** of last contact has a tangent line t. If t is the same as l then we are done. If t is different from l, then pick a tolerance τ so that in every f.o.v. the angle between t and l can be seen. Then with this tolerance zoom in