CHAPTER XVI.

A DIRECT METHOD OF OBTAINING THE EQUATIONS CONNECTING ϑ-PRODUCTS.

290. THE result given as Ex. xi. of § 286, in the last chapter, is a particular case of certain equations which may be obtained by actually multiplying together the theta series and arranging the product in a different way. We give in this chapter three examples of this method, of which the last includes the most general case possible. The first two furnish an introduction to the method and are useful for comparison with the general theorem. The theorems of this chapter do not require the characteristics to be half-integers.

291. *Lemma.* If b be a symmetrical matrix of p^2 elements, $U, V, u, v, A, B, f, g, q, r, f', g', q', r', M, N, s', t', m, n$ be columns, each of p elements, subject to the equations

$$n + m = 2N + s', \quad q' + r' = f', \quad q + r = f, \quad U + V = 2u = A,$$
$$-n + m = 2M + t', \quad -q' + r' = g', \quad -q + r = g, \quad -U + V = 2v = B,$$

then

$$2U(n+q') + b(n+q')^2 + 2\pi i q(n+q') + 2V(m+r') + b(m+r')^2 + 2\pi i r(m+r')$$
$$= 2A\left(N + \frac{s'+f'}{2}\right) + 2b\left(N + \frac{s'+f'}{2}\right)^2 + 2\pi i f\left(N + \frac{s'+f'}{2}\right)$$
$$+ 2B\left(M + \frac{t'+g'}{2}\right) + 2b\left(M + \frac{t'+g'}{2}\right)^2 + 2\pi i g\left(M + \frac{t'+g'}{2}\right).$$

This the reader can easily verify.

Suppose now that the elements of s' and t' are each either 0 or 1, and that n and m take, independently, all possible positive and negative integer values. To any pair of values, the equations $n + m = 2N + s'$, $-n + m = 2M + t'$ give a corresponding pair of values for integers N and M, and a pair of values for s' and t'. Since $2m = 2N + 2M + s' + t'$, $s' + t'$ is even, and therefore, since each element of s' and t' is < 2, s' must be equal to t'. Hence by means of the 2^p possible values for s', the pairs (n, m) are divisible into 2^p sets, each characterised by a certain value of s'. Conversely to any assignable