# CHAPTER XIV.

## FACTORIAL FUNCTIONS.

252. THE present chapter is concerned* with a generalisation of the theory of rational functions and their integrals. As in that case, it is convenient to consider the integrals and the functions together from the first. In order, therefore, that the reader may be better able to follow the course of the argument, it is desirable to explain, briefly, at starting, the results obtained. All the functions and integrals considered have certain fixed singularities, at places† denoted by $c_1, \dots, c_k$. A function or integral which has no infinities except at these fixed singularities is described as everywhere finite. The functions of this theory which replace the rational functions of the simpler theory have, beside the fixed singularities, no infinities except poles. But the functions differ from rational functions in that their values are not the same at the two sides of any period loop; these values have a ratio, described as the *factor*, which is constant along the loop; and a system of functions is characterised by the values of its factors. We consider two sets of factors, and, correspondingly, two sets of *factorial functions*, those of the *primary system* and those of the *associated system*; their relations are quite reciprocal. We have then a circumstance to which the theory of rational functions offers no parallel; *there may be everywhere finite factorial functions*‡. The number of such functions of the primary system which are linearly independent is denoted by $\sigma' + 1$; the number of the associated system by $\sigma + 1$. As in the case of algebraical integrals, we may have *everywhere finite factorial integrals*. The number of such integrals of the primary system which are linearly independent is denoted by $\varpi$, that of the associated system by $\varpi'$. The factorial integrals of the primary system are not integrals of factorial functions of that system; they are chosen so that the values $u$, $u'$

* The subject of the present chapter has been considered by Prym, *Crelle*, LXX. (1869), p. 354; Appell, *Acta Mathematica*, XIII. (1890); Ritter, *Math. Annal.* XLIV. (1894), pp. 261—374. In these papers other references will be found. See also Hurwitz, *Math. Annal.* XLI. (1893), p. 434, and, for a related theory, not considered in the present chapter, Hurwitz, *Math. Annal.* XXXIX. (1891), p. 1. For the latter part of the chapter see the references given in §§ 273, 274, 279.

† In particular the theory includes the case when $k=0$, and no such places enter.

‡ This statement is made in view of the comparison instituted between the development of the theory of rational functions and that of factorial functions. The factorial functions have (unless $k=0$) fixed infinities.