CHAPTER XIII.

ON RADICAL FUNCTIONS.

240. THE reader is already familiar with the fact that if $\operatorname{sn} u$ represent the ordinary Jacobian elliptic function, the square root of $1-\operatorname{sn}^2 u$ may be treated as a single-valued function of u. Such a property is possessed by other square roots. Thus for instance we have*

$$\sqrt{(1-\operatorname{sn} u)(1-k\operatorname{sn} u)}$$
$$= M \sin\frac{\pi}{4K}(K-u)\prod_m \frac{\left[1-2q^m\sin\frac{\pi u}{2K}+q^{2m}\right]\left[1-2q^{m-\frac{1}{2}}\sin\frac{\pi u}{2K}+q^{2m-1}\right]}{1-2q^{2m-1}\cos\frac{\pi u}{K}+q^{4m-2}},$$

where M is a certain constant, and, as usual, $q=e^{-\pi K'/K}$. The single-valuedness of the function $\sqrt{(1-\operatorname{sn} u)(1-k\operatorname{sn} u)}$ can be immediately seen to follow from the fact that *each of the zeros and poles of the function* $(1-\operatorname{sn} u)(1-k\operatorname{sn} u)$ *is of the second order.* It is manifest that we can easily construct other functions having the same property. If now we write $u=u^{x,a}$ and consider the square root on the dissected elliptic Riemann surface, we shall thereby obtain a single-valued function of the place x, whose values on the two sides of either period loop will have a ratio, constant along that loop, which is equal to ± 1.

Ex. Prove that the function

$$\sqrt{(\sqrt{\wp u-e_1}-\sqrt{e_2-e_1})(\sqrt{\wp u-e_1}-\sqrt{e_3-e_1})}$$

is a single-valued function of u.

Further we have, in Chapter XI., in dealing with the hyperelliptic case associated with an equation of the form

$$y^2=(x-a_1)\dots(x-a_{2p})(x-c),$$

* Cf. Cayley, *Elliptic Functions* (1876), Chap. XI. The function may be regarded as a doubly periodic function, with $8K$, $2iK'$ as its fundamental periods. It is of the fourth order, with K, $5K$, $K+iK'$, $5K+iK'$ as zeros, and iK', $2K+iK'$, $4K+iK'$, $6K+iK'$ as poles.