CHAPTER VIII.

ABEL'S THEOREM; ABEL'S DIFFERENTIAL EQUATIONS.

148. THE present chapter is mainly concerned with that theorem with which the subject of the present volume may be said to have begun. It will be seen that with the ideas which have been analysed in the earlier part of the book, the statement and proof of that theorem is a matter of great simplicity.

The problem of the integration of a rational algebraical function (of a single variable) leads to the introduction of a transcendental function, the logarithm; and the integral of any such rational function can be expressed as a sum of rational functions and logarithms of rational functions. More generally, an integral of the form

$$\int dx\, R(x, y, y_1, \ldots, y_k),$$

wherein $x, y, y_1, y_2, \ldots$ are capable of rational expression in terms of a single parameter, and R denotes any rational algebraic function, can be expressed as a sum of rational functions of this parameter, and logarithms of rational functions of the same. This includes the case of an integral of the form

$$\int dx\, R(x, \sqrt{ax^2+bx+c}).$$

But an integral of the form

$$\int dx\, R(x, \sqrt{ax^4+bx^3+cx^2+dx+e})$$

cannot, in general, be expressed by means of rational or logarithmic functions; such integrals lead in fact to the introduction of other transcendental functions than the logarithm, namely to elliptic functions; and it appears that the nearest approach to the simplicity of the case, in which the subject of integration is a rational function, is to be sought in the relations which exist for the *sums* of like elliptic integrals. For instance, we have the equation

$$\int_0^{x_1} \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}} + \int_0^{x_2} \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}} - \int_0^{x_3} \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}} = 0,$$