CHAPTER V.

On certain forms of the Fundamental Equation of the Riemann Surface.

51. We have already noticed that the Riemann surface can be expressed in many different ways, according to the rational functions used as variables. In the present chapter we deal with three cases: the first, the hyperelliptic case (§§ 51—59), is a special case, and is characterised by the existence of a rational function of the second order; the second, which we shall often describe as that of Weierstrass's canonical surface (§§ 60—68), is a general case obtained by choosing, as independent variables, two rational functions whose poles are at one place of the surface: the third case referred to (§§ 69—71) is also a general case, which may be regarded as a generalization of the second case. It will be seen that both the second and third cases involve ideas which are in close connexion with those of the previous chapter. The chapter concludes with an account of a method for obtaining the fundamental integral functions for any fundamental algebraic equation whatever (§§ 73—79).

It may be stated for the guidance of the reader that the results obtained for the second and third cases (§§ 60—71) are not a necessary preliminary to the theory of the remainder of the book; but they will be found to furnish useful examples of the actual application of the theory.

52. We have seen that when p is greater than zero, no rational function of the first order exists. We consider now the consequences of the hypothesis of the existence of a rational function of the second order. Let ξ denote such a function; let c be any constant and α, β denote the two places where $\xi = c$, so that $(\xi - c)^{-1}$ is a rational function of the second order with poles at α, β. The places α, β cannot coincide for all values of c, because the rational function $d\xi/dx$ has only a finite number of zeros. We may therefore regard α, β as distinct places, in general. The most general rational function which has simple poles at α, β cannot contain more than two linearly entering arbitrary constants. For if such a function be $\lambda + \lambda_1 f_1 + \lambda_2 f_2 + \dots$, λ, λ_1, ... being arbitrary constants, each of the functions f_1, f_2, ... must be of the second order at most and therefore actually of the second order: by choosing the constants so that the sum of the residues at α is zero, we can therefore