CHAPTER XIX

ELLIPTIC FUNCTIONS AND INTEGRALS

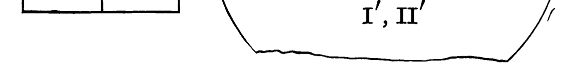
187. Legendre's integral I and its inversion. Consider

$$z = \int_0^w \frac{dw}{\sqrt{(1-w^2)(1-k^2w^2)}}, \qquad 0 < k < 1. \tag{I}$$

The Riemann surface for the integrand* has branch points at $w = \pm 1$ and $\pm 1/k$ and is of two sheets. Junction lines may be drawn between $+1$, $+1/k$ and -1, $-1/k$. For very large values of w, the radical $\sqrt{(1-w^2)(1-k^2w^2)}$ is approximately $\pm kw^2$ and hence there is no danger of confusing the values of the function. Across the junction lines the surface may be connected as indicated, so that in the neighborhood of $w = \pm 1$ and $w = \pm 1/k$ it looks like the surface for $\sqrt{w}$. Let $+1$ be the value of the integrand at $w = 0$ in the upper sheet. Further let

$$K = \int_0^1 \frac{dw}{\sqrt{(1-w^2)(1-k^2w^2)}}, \qquad iK' = \int_1^{\frac{1}{k}} \frac{dw}{\sqrt{(1-w^2)(1-k^2w^2)}}. \tag{1}$$

Let the changes of the integral be followed so as to map the surface on the z-plane. As w moves from o to a, the integral (I) increases by K, and z moves from O to A. As w continues straight on, z makes a right-angle turn and increases by pure imaginary increments to the total amount iK' when w reaches b. As w continues there is another right-angle turn in z, the integrand again becomes real, and z moves down to C. (That z reaches C follows from the facts that the



z—plane *w—surface*

* The reader unfamiliar with Riemann surfaces (§ 184) may proceed at once to identify (I) and (2) by Ex. 9, p. 475 and may take (1) and other necessary statements for granted.