PART I. DIFFERENTIAL CALCULUS

CHAPTER III

TAYLOR'S FORMULA AND ALLIED TOPICS

31. Taylor's Formula. The object of Taylor's Formula is to express the value of a function $f(x)$ in terms of the values of the function and its derivatives at some one point $x = a$. Thus

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \cdots + \frac{(x-a)^{n-1}}{(n-1)!}f^{(n-1)}(a) + R. \tag{1}$$

Such an expansion is necessarily true because the remainder R may be considered as defined by the equation; the real significance of the formula must therefore lie in the possibility of finding a simple expression for R, and there are several.

THEOREM. On the hypothesis that $f(x)$ and its first n derivatives exist and are continuous over the interval $a \leqq x \leqq b$, the function may be expanded in that interval into a polynomial in $x - a$,

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \cdots + \frac{(x-a)^{n-1}}{(n-1)!}f^{(n-1)}(a) + R, \tag{1}$$

with the remainder R expressible in any one of the forms

$$R = \frac{(x-a)^n}{n!}f^{(n)}(\xi) = \frac{h^n(1-\theta)^{n-1}}{(n-1)!}f^{(n)}(\xi) = \frac{1}{(n-1)!}\int_0^h t^{n-1}f^{(n)}(a+h-t)\,dt, \tag{2}$$

where $h = x - a$ and $a < \xi < x$ or $\xi = a + \theta h$ where $0 < \theta < 1$.

A first proof may be made to depend on Rolle's Theorem as indicated in Ex. 8, p. 49. Let x be regarded for the moment as constant, say equal to b. Construct