Chapter 24
3-MANIFOLDS — THE SHAPE OF SPACE


Jeff Weeks, Reuben Hersh, and David Henderson at Fall 2002 MAA Seaway Section meeting

> ... if we look at the extreme points of the sky, all the visual rays appear equal to us, and if diametrically opposed stars describe a great circle, one is setting while the other is rising. If the universe, instead of being spherical, were a cone or a cylinder, or a pyramid or any other solid, it would not produce this effect on earth: one of its parts would appear larger, another smaller, and the distances from earth to heaven would appear unequal. — Theon of Smyrna (~70–~135, Greek), [AT: Theon]

> It will be shown that a multiply extended quantity [three-dimensional manifold] is susceptible of various metric relations, so that Space constitutes only a special case of a triply extended quantity. From this however it is a necessary consequence that the theorems of geometry cannot be deduced from general notions of quantity, but that those properties that distinguish Space from other conceivable triply extended quantities can only be inferred from experience. — G. F. B. Riemann (1826–1866, German) On the Hypotheses Which Lie at the Foundations of Geometry, translated in [DG: Spivak], Vol. II, p. 135.

Now we come to where we live. We live in a physical 3-dimensional space, that is locally like Euclidean 3-space. The fundamental question we will investigate in this chapter is, how can we tell what the shape of our Universe is? This is the same question both Theon of Smyrna and Riemann were attempting to answer in the above quotes. The question about the shape of the Universe is the question about two geometries: local (primarily described by its curvature) and global (topology describing general global properties of the shape of Universe as of a continuous object). The shape of the Universe is related to the theory of general relativity, describing curved spacetime.