Chapter 20

TRIGONOMETRY AND DUALITY

After we have found the equations [The Laws of Cosines and Sines for a Hyperbolic Plane] which represent the dependence of the angles and sides of a triangle; when, finally, we have given general expressions for elements of lines, areas and volumes of solids, all else in the [Hyperbolic] Geometry is a matter of analytics, where calculations must necessarily agree with each other, and we cannot discover anything new that is not included in these first equations from which must be taken all relations of geometric magnitudes, one to another. ... We note however, that these equations become equations of spherical Trigonometry as soon as, instead of the sides a, b, c we put ... a $\sqrt{-1}$,b $\sqrt{-1}$,c $\sqrt{-1}$ …— N. *Lobachevsky, quoted in [***HY***: Greenberg]*

In this chapter, we will first derive, geometrically, expressions for the circumference of a circle on a sphere, the Law of Cosines on the plane, and its analog on a sphere. Then we will talk about duality on a sphere. On a sphere, duality will enable us to derive other laws that will help our two-dimensional bug to compute sides and/or angles of a triangle given ASA, RLH, SSS, or AAA. Finally, we will look at duality on the plane.

PROBLEM 20.1 CIRCUMFERENCE OF A CIRCLE

a. *Find a simple formula for the circumference of a circle on a sphere in terms of its intrinsic radius and make the formula as intrinsic as possible.*

We suggest that you make an extrinsic drawing (like Figure 20.1) of the circle, its intrinsic radius, its extrinsic radius, and the center of the sphere. You may well find it convenient to use trigonometric functions to express your answer. Note that the existence of trigonometric functions for right triangles follows from the properties of similar triangles that were proved in Problem **13.4**.