Chapter 5

Area, Parallel Transport and Intrinsic Curvature

In Chapters 3 and 4 we were developing extrinsic descriptions of the intrinsic curvature of a curve on a surface. In this chapter we will develop intrinsic descriptions of both the ***intrinsic curvature of a curve*** on a surface and the ***intrinsic curvature of a surface***. This intrinsic curvature will be developed by investigating the relationships between surface area, normal curvature, and ***parallel transport***, a notion of local parallelism that is definable on all surfaces. We first develop these connections on a sphere that has known curvature and then use the results on the sphere to motivate the discussion on general surfaces. As an important part of this development, we introduce the notion of ***holonomy***, which also has many other applications in modern differential geometry and engineering.

PROBLEM 5.1. The Area of a Triangle on a Sphere

DEFINITION: On any surface we will call a triangle a ***geodesic triangle*** if its three sides are geodesic segments.

a. *Let Δ be a geodesic triangle on a sphere. Show that the formula*

$$\text{Area}\ (\Delta) = A/4\pi[\ \sum \beta_i - \pi\ ] = A/4\pi[\ 2\pi - \sum \alpha_i\ ]$$

holds, where A is the area of the sphere, β_i are the interior angles, and α_i are the exterior angles of the triangle in radians. The quantity $(\Sigma\beta_i)-\pi$ is called the ***excess***.

We offer the following hint as a way to approach this problem: Find the area of a biangle (lune) with angle θ. (A ***biangle*** or ***lune*** is one of the connected regions between two great circles.) Notice that the great circles that contain the sides of the triangle divide the sphere into overlapping biangles. Focus on the biangles determined by either the interior angles or the exterior angles.

This is one of the problems that you almost certainly must do on an actual sphere. There are too many things to see, and the drawings we make on paper distort lines and angles too much. The best way to start is to make a small triangle on a sphere, and extend the lines of the triangle to complete great circles. Then look at the results. You will find an identical triangle on the other side of the sphere, and you can see several lunes extending out from the triangles. The key to this problem is to put everything in terms of areas that you know. *Find the areas of the lunes*, as noted above. After that, it is simply a matter of adding everything up properly.

We know that the area of the whole sphere is $4\pi R^2$, where R is the (extrinsic) radius of the sphere. With this additional information we can rewrite the formula of Problem **5.1.a**:

$$\text{Area}\ (\Delta) = [\ \sum \beta_i - \pi\ ]\ R^2 = [\ 2\pi - \sum \alpha_i\ ]\ R^2.$$

b. *Use* **5.1.a** *to show that for a triangle in the plane the sum of the interior angles is π and the sum of the exterior angles is 2π.*