Chapter 3

Extrinsic Descriptions of Intrinsic Curvature

When we view a curve on a smooth surface, then we can talk about its ***intrinsic curvature*** (sometimes called ***geodesic curvature***) with respect to the surface. As in Chapter 2, we want the curvature to be the rate of change of the tangent direction (with respect to arc length), except now we want to look at those changes that are intrinsic to the surface (that is, the changes that a 2-dimensional bug on the surface would be able to detect by perceiving only what is on the surface). The notions from Problem **2.3** can be used, but we have to be careful. For example, the machinery of ordinary linear algebra does not directly allow us to compare tangent vectors at two different points on the surface. This is because the vectors tangent to the surface do not, in general, form a vector space, since the difference of two vectors tangent to a surface at different points is not necessarily also a vector tangent to the surface. *Check this out on small portions of a cylinder and sphere.* (See discussion before Problem **2.2**.) We don't know how to talk about vectors at two different points on a sphere as being the "same". North/South/East/West terminology does not work on the sphere because these directions depend on the choice of pole and because at the north pole every direction is south.

On a surface, a curve with no intrinsic curvature is said to be ***intrinsically straight*** (with respect to that surface) or is called a ***geodesic***. These are the curves that a 2-dimensional bug would experience as straight. Along a geodesic, the tangent direction is not changing intrinsically, but of course, in general, it is changing extrinsically. So, we want to be able to find a way of talking about two tangent directions being equal intrinsically (along a curve) when they are not necessarily equal extrinsically — this we will do in Chapter 5.

In this chapter we give extrinsic descriptions of intrinsic curvature. These descriptions make sense to us viewing the surface extrinsically but are of no use to the 2-dimensional bug. Nor will they be useful to us in our experience as intrinsic "bugs" in our 3-dimensional physical universe where we have no extrinsic experience. We will remedy this situation later in Chapter 5 where we will finally arrive at intrinsic descriptions.

PROBLEM 3.1. Smooth Surfaces and Tangent Planes

An (***extrinsic***) ***smooth surface***, M, is a geometric figure in $\mathbf{R}^n$ that is uniformly infinitesimally planar (or flat). We say that a surface is *infinitesimally planar* at the point p in M if, when you zoom in on p, then M will become indistinguishable from a plane, $T_p M$, called the ***tangent plane at*** p; that is, for every tolerance $\tau > 0$, there is a δ, such that in any f.o.v. centered at p of radius $< \delta$, it is the case that the projection of M onto $T_p M$ is one-to-one and moves points less than $\tau\delta$ in the f.o.v. [Remember that the tolerance is a percentage such that two points are indistinguishable (in the f.o.v.) if their distance apart is less than the tolerance times the diameter of the f.o.v. (See Problem **2.1**.)] The surface is uniformly infinitesimally planar if there is some neighborhood of p such that (for each tolerance τ) the same δ can be used for each point in the neighborhood. Computer Exercise 3.1 may be used to view a surface given parametrically and its tangent plane at a specified point. (See Problem **4.1** for further discussion of the tangent plane.)