Problem 3.1. Smooth Surfaces and Tangent Planes

*a.*

If the surface is infinitesimally planar at \( p \) and the tangent planes vary continuously, then, for every tolerance \( \tau/4 \) there is a field of view with center \( p \) and radius \( \rho \), such that, if \( p \) and \( x \) are both on the surface in the field of view, then each point on the surface is within \( \tau \rho/4 \) of the tangent plane \( T_p \) and each point on \( T_p \) is within \( \tau \rho/4 \) of the tangent plane \( T_x \). Then, for every \( q \) on the surface such that \( |p - q| < \rho/2 \), every point on the surface within \( \rho/2 \) of \( q \) is within \( \tau \rho/4 \) of \( T_p \) which in turn is within \( \tau \rho/4 \) of \( T_q \). Thus every point on the surface within \( \rho/2 \) of \( q \) is within \( \tau \rho/2 \) of the tangent plane \( T_q \). Thus continuously infinitesimally planar implies smooth (uniformly infinitesimally planar).

*b.*

If the partial derivatives are linearly independent then they span a plane which is the tangent plane. Since the partial derivatives vary continuously the tangent planes must also vary continuously and thus by Part a the surface is smooth.

The function \( x(x,y) = (x^3,y) \) is a coordinate patch for the plane but the partial derivatives with respect to \( x \) are zero when \( x = 0 \) and thus, the partial derivatives are not linearly independent.

c.

One may use the coordinate patches for these surfaces from Chapter 1 and check that the partial derivatives are linearly independent, using Part b. Or, one may argue geometrically that each of these surfaces has tangent planes and for every tolerance \( \tau \) the same \( \rho \) will work for all points except on the cone. On the cone there are obvious tangent planes which are tangent to the cone along a generator and for every tolerance \( \tau \) the amount of zooming necessary is dependent (in a linear fashion) on how far the point is from the cone point, and thus, the amount of zooming is uniform over neighborhoods whose closures miss the cone point.

d.

If the function \( f \) is smooth then the partial derivatives are \( x_1(\theta,x) = (1, f'(x) \cos \theta, f'(x) \sin \theta) \) and \( x_2(\theta,x) = (0, -f(x) \sin \theta, f(x) \cos \theta) \), which vary continuously and are nonzero and perpendicular (and thus linearly independent) as long as \( f(x) \) is not zero. Thus, by Part b the surface is smooth.

If the surface is smooth, then the tangent planes along the curve \( \theta = 0 \) are perpendicular to the plane \( \theta = 0 \) and thus intersect that in lines which are tangent to the graph of \( f \). Thus, by 2.2, \( f \) is continuously differentiable.