Appendix A

Linear Algebra from a Geometric Point of View

Whoever thinks algebra is a trick in obtaining unknowns has thought it in vain. No attention should be paid to the fact that algebra and geometry are different in appearance. Algebras (al-jabbre and maqabeleh) are geometric facts which are proved by Propositions Five and Six of Book Two of [Euclid's] Elements.
— Omar Khayyam, a paper [**A:** Khayyam (1963)]

A.0. Where Do We Start?

Usual treatments of linear and affine algebra start with a vector space as a set of "vectors" and the operations of vector addition and scalar multiplication that satisfy the axioms for a vector space. In a vector space all vectors emanate from the origin. This works well algebraically; but it ignores our geometric images and experiences of vectors.

Geometrically, we start with our experiences of Euclidean geometry where there is no point that has been singled out as the origin and where there are no numerical distances (until after a unit distance is chosen).

The linear structure of Euclidean space is carried by the translations of the space. We picture vectors as directed line segments from one point to another, and the translations serve to define when one vector is parallel to another. Any space whose translations satisfy the same properties as translations in Euclidean space is called a ***geometric affine space***, which is the subject of Section **A.1**.

The collection of all the vectors emanating from the same point is called the tangent space at that point. Using translations we may define addition and scalar multiplication of vectors. The properties of these two operations on vectors are the defining properties of a ***vector space***, which is the subject of Section **A.2**.

A.1. Geometric Affine Spaces

A ***geometric affine space over the field*** K is a space, S, together with bijections, $T_{ba}: S \to S$, $T_{ba}(a) = b$, which, for every pair of points a, b in S, satisfy the Properties (0)-(8), below. In this text K will always be the field of real numbers **R**. We call T_{ba} ***the translation from*** a ***to*** b. We call the ordered pair (a,b) ***the (bound) vector from*** a ***to*** b. The most basic property of translations is:

(0) T_{ba} is unique in the sense that, if $T_{dc}(a) = b$, then $T_{ba} = T_{dc}$; and translations are closed under composition in the sense that

$$T_{ba}T_{dc} = T_{ec}, \text{ where } e = T_{ba}(d)\text{ , where } (AB)(x) = A(B(x)).$$

Further, we assume that T_{aa} = identity [that is, $T_{aa}(x) = x$, for all x in S.]

[Note the implication that, if a is distinct from b, then T_{ba} has no fixed points.]

This property allows us to define when two bound vectors are equivalent. We say that (a,b) ***is parallel to*** (c,d) if there is a translation that takes (a,b) to (c,d), in symbols $(a,b)\approx(c,d)$. Property (0) assures us