Chapter 8

# Intrinsic Local Descriptions and Manifolds

In Chapter 5 we developed geometrically intrinsic descriptions of holonomy, parallel transport, and curvature of surfaces. In Chapter 6 we developed extrinsic descriptions of Gaussian curvature and showed that it was the same as the intrinsic curvature for all $C^2$ surfaces. In Chapter 7, we found intrinsic local descriptions of Gaussian (intrinsic) curvature with respect to extrinsically defined local coordinates, using (extrinsic) directional derivatives. Now, in this chapter we will develop an intrinsic directional derivative that will allow intrinsic local descriptions of parallel transport. Then we will introduce the notion of manifolds that may have only intrinsically defined local coordinates. We will then put this all together to find for manifolds intrinsic local descriptions of the important intrinsic notions: covariant derivatives, geodesics, parallel transport, holonomy, Gaussian curvature, and others.

## *PROBLEM 8.1. Covariant Derivative and Connection*

If $\mathbf{X}_p$ is a tangent vector at the point $p$ in $M$, and $\mathbf{f}$ is a vector field (a function that gives a tangent vector at each point) defined near $p$, then the directional derivative $\mathbf{X}_p\mathbf{f}$ is not in general a tangent vector and, thus, is not intrinsic. But we can define an intrinsic directional derivative by slightly modifying the definition of $\mathbf{X}_p\mathbf{f}$. In particular, if $\alpha(t)$ is a curve in $M$ with $\alpha(0) = p$ and $\alpha'(0) = \mathbf{X}_p$, then

$$\mathbf{X}_p\mathbf{f} = \lim_{\delta\to 0} \tfrac{1}{\delta}[\mathbf{f}(\alpha(\delta)) - \mathbf{f}(p)].$$

This fails to be intrinsic only in the vector subtraction

$$[\mathbf{f}(\alpha(\delta)) - \mathbf{f}(p)].$$

Even in Euclidean space this subtraction does not literally make sense, because $\mathbf{f}(\alpha(\delta))$ is a (free) vector with base at the point $\alpha(\delta)$ ,and $\mathbf{f}(p)$ is a (free) vector with base at $p$. So in Euclidean space we perform the subtraction by first parallel translating $\mathbf{f}(\alpha(\delta))$ to a (bound) vector $\mathbf{f}(\alpha(\delta))_p$ based at $p$. (See Figure 8.1.) We can more correctly define

$$\mathbf{X}_p\mathbf{f} = \lim_{\delta\to 0} \tfrac{1}{\delta}\left[\mathbf{f}(\alpha(\delta))_p - \mathbf{f}(p)_p\right].$$

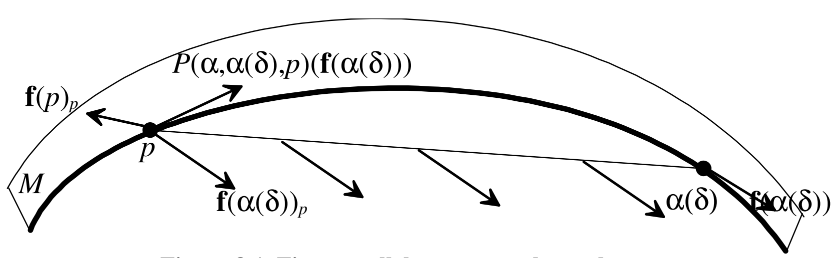


**Figure 8.1. First parallel transport, then subtract.**