Chapter 7  
Applications of Gaussian Curvature

In this chapter we will use the hard-won result from Chapter 6 to express Gaussian (intrinsic) curvature in local coordinates and to find several intrinsic descriptions of Gaussian curvature. Along the way, we will investigate the exponential map and finally come to some resolution concerning the tension between shortest and straight.

**Problem 7.1. Gaussian Curvature in Local Coordinates**

In local coordinates \(x\), the second fundamental form

\[
\II(X, Y) = \II(X^1 x_1 + X^2 x_2, Y^1 x_1 + Y^2 x_2)
\]

can be written as:

\[
\II\left( \begin{pmatrix} X^1 \\ X^2 \end{pmatrix}, \begin{pmatrix} Y^1 \\ Y^2 \end{pmatrix} \right) = \begin{pmatrix} X^1 & X^2 \\ \langle x_{11}, n \rangle & \langle x_{12}, n \rangle \\ \langle x_{21}, n \rangle & \langle x_{22}, n \rangle \end{pmatrix} \begin{pmatrix} Y^1 \\ Y^2 \end{pmatrix}.
\]

Now express the \textbf{unit} principal directions in these coordinates:

\[
T_1 = \begin{pmatrix} T^1_1 \\ T^1_2 \end{pmatrix} \quad \text{and} \quad T_2 = \begin{pmatrix} T^2_1 \\ T^2_2 \end{pmatrix}.
\]

Since (from Problem 6.2),

\[
T_1 n = -\kappa_1 T_1 \quad \text{and} \quad T_2 n = -\kappa_2 T_2,
\]

we have

\[
\II(T_1, T_1) = \kappa_1, \quad \II(T_2, T_2) = \kappa_2, \quad \text{and} \quad \II(T_1, T_2) = \II(T_2, T_1) = 0.
\]

Thus, we can see that:

\[
\begin{pmatrix} T^1_1 & T^2_1 \\ T^1_2 & T^2_2 \end{pmatrix} \begin{pmatrix} \langle x_{11}, n \rangle & \langle x_{12}, n \rangle \\ \langle x_{21}, n \rangle & \langle x_{22}, n \rangle \end{pmatrix} \begin{pmatrix} T^1_1 \\ T^1_2 \\ T^2_1 \\ T^2_2 \end{pmatrix} = \begin{pmatrix} \kappa_1 & 0 \\ 0 & \kappa_2 \end{pmatrix}.
\]

Thus (using the result from matrix algebra that the determinant of a product is the product of the determinants):

\[
K = \kappa_1 \kappa_2 = \det \begin{pmatrix} \kappa_1 & 0 \\ 0 & \kappa_2 \end{pmatrix} = \det\begin{pmatrix} T^1_1 & T^2_1 \\ T^1_2 & T^2_2 \end{pmatrix} \det\begin{pmatrix} \langle x_{11}, n \rangle & \langle x_{12}, n \rangle \\ \langle x_{21}, n \rangle & \langle x_{22}, n \rangle \end{pmatrix} \det\begin{pmatrix} T^1_1 & T^2_1 \\ T^1_2 & T^2_2 \end{pmatrix}.
\]