Chapter 7

Applications of Gaussian Curvature

In this chapter we will use the hard-won result from Chapter 6 to express Gaussian (intrinsic) curvature in local coordinates and to find several intrinsic descriptions of Gaussian curvature. Along the way, we will investigate the exponential map and finally come to some resolution concerning the tension between *shortest* and *straight*.

PROBLEM 7.1. Gaussian Curvature in Local Coordinates

In local coordinates **x**, the second fundamental form

$$\mathrm{II}(\mathbf{X},\mathbf{Y}) = \mathrm{II}(X^1\mathbf{x}_1+X^2\mathbf{x}_2,\ Y^1\mathbf{x}_1+Y^2\mathbf{x}_2)$$

can be written as:

$$\mathrm{II}\left(\begin{pmatrix} X^1 \\ X^2 \end{pmatrix},\begin{pmatrix} Y^1 \\ Y^2 \end{pmatrix}\right) = \begin{pmatrix} X^1 & X^2 \end{pmatrix}\begin{pmatrix} \langle \mathbf{x}_{11},\mathbf{n}\rangle & \langle \mathbf{x}_{12},\mathbf{n}\rangle \\ \langle \mathbf{x}_{21},\mathbf{n}\rangle & \langle \mathbf{x}_{22},\mathbf{n}\rangle \end{pmatrix}\begin{pmatrix} Y^1 \\ Y^2 \end{pmatrix}.$$

Now express the **unit** principal directions in these coordinates:

$$\mathbf{T}_1 = \begin{pmatrix} T_1^1 \\ T_1^2 \end{pmatrix} \text{ and } \mathbf{T}_2 = \begin{pmatrix} T_2^1 \\ T_2^2 \end{pmatrix}.$$

Since (from Problem **6.2**),

$$\nabla_{\mathbf{T}_1}\mathbf{n} = -\kappa_1\mathbf{T}_1 \text{ and } \nabla_{\mathbf{T}_2}\mathbf{n} = -\kappa_2\mathbf{T}_2,$$

we have

$$\mathrm{II}(\mathbf{T}_1,\mathbf{T}_1) = \kappa_1,\ \mathrm{II}(\mathbf{T}_2,\mathbf{T}_2) = \kappa_2, \text{ and } \mathrm{II}(\mathbf{T}_1,\mathbf{T}_2) = \mathrm{II}(\mathbf{T}_2,\mathbf{T}_1) = 0.$$

Thus, we can see that:

$$\begin{pmatrix} T_1^1 & T_1^2 \\ T_2^1 & T_2^2 \end{pmatrix}\begin{pmatrix} \langle \mathbf{x}_{11},\mathbf{n}\rangle & \langle \mathbf{x}_{12},\mathbf{n}\rangle \\ \langle \mathbf{x}_{21},\mathbf{n}\rangle & \langle \mathbf{x}_{22},\mathbf{n}\rangle \end{pmatrix}\begin{pmatrix} T_1^1 & T_2^1 \\ T_1^2 & T_2^2 \end{pmatrix} = \begin{pmatrix} \kappa_1 & 0 \\ 0 & \kappa_2 \end{pmatrix}.$$

Thus (using the result from matrix algebra that the determinant of a product is the product of the determinants):

$$K = \kappa_1\kappa_2 = \det\begin{pmatrix} \kappa_1 & 0 \\ 0 & \kappa_2 \end{pmatrix} =$$

$$= \det\begin{pmatrix} T_1^1 & T_1^2 \\ T_2^1 & T_2^2 \end{pmatrix}\det\begin{pmatrix} \langle \mathbf{x}_{11},\mathbf{n}\rangle & \langle \mathbf{x}_{12},\mathbf{n}\rangle \\ \langle \mathbf{x}_{21},\mathbf{n}\rangle & \langle \mathbf{x}_{22},\mathbf{n}\rangle \end{pmatrix}\det\begin{pmatrix} T_1^1 & T_2^1 \\ T_1^2 & T_2^2 \end{pmatrix}.$$