Chapter 6

Gaussian Curvature Extrinsically Defined

Pep Talk to the Reader

I think that the material in this chapter is very difficult. Don't give up and don't lose hope. What is happening here is that we are standing at the interface between things that we can see and argue about geometrically, and things that are given formally. It is difficult to hold these two aspects together—one is alive and one is dead, but both are important. It sometimes feels easier to jump headlong into the formal stuff—forgetting about what it means geometrically and just following everything through mechanically. Don't do that! Unfortunately that is often the tendency—it is also my tendency! Resist it and persevere in trying to see what the meanings of these formal things are geometrically as you go along. This is hard to do, but the effort will be well worth it. On the other hand, there exists a tendency to ignore the formal stuff and rely only on our geometric intuition. But, if we ignore the formal stuff, we would miss out on the incredibly powerful tools contained in the formalism. We need to use both the formal analytic tools and our geometric intuition; and we need to look for their interrelations. Relate everything in this chapter to the example of surfaces you already know, such as the sphere, cylinder, cone, ribbon, and strake.

In Chapter 5 we developed an intrinsic description of the intrinsic curvature of a surface. In this chapter we start with the more common extrinsic description of the Gaussian curvature of a surface, which is based on the normal curvature introduced in Problem **4.7.a**. The Gaussian and intrinsic curvatures are easily seen to be the same on a sphere. Then we use a mapping (called the ***Gauss map***) from the surface to the sphere, which then allows us to show that the Gaussian curvature and intrinsic curvature coincide on all C^2 surfaces.

In Chapter 7 we will use these results to express the Gaussian (intrinsic) curvature in local coordinates and to derive several more intrinsic descriptions of Gaussian curvature.

At the end of this chapter we will explore *mean curvature* and *minimal surfaces*.

Problem 6.1. Gaussian Curvature, Extrinsic Definition

Let p be a point on the smooth C^2 surface M in $\mathbf{R}^3$, and let $\mathbf{n}(p)$ be one of the two choices of unit normal to the surface at p, so that $\mathbf{n}$ is differentiable in a neighborhood of p. Let $\mathbf{T}_p$ be a unit tangent vector at p. If γ is a curve on M, which passes through p and has $\mathbf{T}_p$ as unit tangent vector, then, according to Problem **4.7.a**, the normal curvature of γ at p satisfies

$$\boldsymbol{\kappa}_\mathbf{n}(p) = \langle \mathbf{T}_p, -\mathbf{T}_p\mathbf{n}\rangle\, \mathbf{n}(p).$$

Since $\mathbf{n}(p)$ is a unit vector, $\langle \mathbf{T}_p, -\mathbf{T}_p\mathbf{n}\rangle$ is the magnitude of the normal curvature vector, and thus we define the (***scalar***) ***normal curvature*** of M at p in the direction $\mathbf{T}_p$ as

$$\kappa_\mathbf{n}(\mathbf{T}_p) \equiv \langle \mathbf{T}_p, -\mathbf{T}_p\mathbf{n}\rangle$$

$$\equiv \text{the length of the projection of } -\mathbf{T}_p\mathbf{n} \text{ onto the direction of } \mathbf{T}_p.$$