CHAPTER XV.

Relations connecting products of theta functions—Introductory.

280. As preparatory to the general theory of multiply-periodic functions of several variables, and on account of the intrinsic interest of the subject, the study of the algebraic relations connecting the theta functions is of great importance. The multiplicity and the complexity of these relations render any adequate account of them a matter of difficulty; in this volume the plan adopted is as follows:—In the present chapter are given some preliminary general results frequently used in what follows, with some examples of their application. The following Chapter (XVI.) gives an account of a general method of obtaining theta relations by actual multiplication of the infinite series. In Chapter XVII. a remarkable theory of groups of half-integer characteristics, elaborated by Frobenius, is explained, with some of the theta relations that result; from these the reader will perceive that the theory is of great generality and capable of enormous development. References to the literature, which deals mostly with the case of half-integer characteristics, are given at the beginning of Chapter XVII.

281. Let $\phi(u_1, \ldots, u_p)$ be a single-valued function of p independent variables $u_1, \ldots, u_p$, such that, if $a_1, \ldots, a_p$ be a set of *finite* values for $u_1, \ldots, u_p$ respectively, the value of $\phi(u_1, \ldots, u_p)$, for any set of finite values of $u_1, \ldots, u_p$, is expressible by a converging series of ascending integral positive powers of $u_1 - a_1, u_2 - a_2, \ldots, u_p - a_p$. Such a function is an integral analytical function. Suppose further that $\phi(u_1, \ldots, u_p)$ has for each of its arguments, independently of the others, the period unity, so that if m be any integer, we have, for $\alpha = 1, 2, \ldots, p$, the equation

$$\phi(u_1, \ldots, u_\alpha + m, \ldots, u_p) = \phi(u_1, \ldots, u_p).$$

Then* the function $\phi(u_1, \ldots, u_p)$ can be expressed by an infinite series of the form

$$\sum_{n_1=-\infty}^{\infty} \cdots\cdots \sum_{n_p=-\infty}^{\infty} A_{n_1, \ldots, n_p} e^{2\pi i (u_1 n_1 + \ldots + u_p n_p)},$$

* For the nomenclature and another proof of the theorem, see Weierstrass, *Abhandlungen aus der Functionenlehre* (Berlin, 1886), p. 159, etc.