CHAPTER XVIII

FUNCTIONS OF A COMPLEX VARIABLE

178. General theorems. The complex function $u(x, y) + iv(x, y)$, where $u(x, y)$ and $v(x, y)$ are single valued real functions continuous and differentiable partially with respect to x and y, has been defined as a function of the complex variable $z = x + iy$ when and only when the relations $u'_x = v'_y$ and $u'_y = -v'_x$ are satisfied (§ 73). In this case the function has a derivative with respect to z which is independent of the way in which Δz approaches the limit zero. Let $w = f(z)$ be a function of a complex variable. Owing to the existence of the derivative the function is necessarily continuous, that is, if ϵ is an arbitrarily small positive number, a number δ may be found so small that

$$|f(z) - f(z_0)| < \epsilon \quad \text{when} \quad |z - z_0| < \delta, \tag{1}$$

and moreover this relation holds uniformly for all points z_0 of the region over which the function is defined, provided the region includes its bounding curve (see Ex. 3, p. 92).

It is further assumed that the derivatives u'_x, u'_y, v'_x, v'_y are continuous and that therefore the derivative $f'(z)$ is continuous.* The function is then said to be an *analytic function* (§ 126). All the functions of a complex variable here to be dealt with are analytic in general, although they may be allowed to fail of being analytic at certain specified points called *singular points*. The adjective "analytic" may therefore usually be omitted. The equations

$$w = f(z) \quad \text{or} \quad u = u(x, y), \qquad v = v(x, y)$$

define a transformation of the xy-plane into the uv-plane, or, briefer, of the z-plane into the w-plane; to each point of the former corresponds one and only one point of the latter (§ 63). If the Jacobian

$$\begin{vmatrix} u'_x & u'_y \\ v'_x & v'_y \end{vmatrix} = (u'_x)^2 + (v'_x)^2 = |f'(z)|^2 \tag{2}$$

* It may be proved that, in the case of functions of a complex variable, the continuity of the derivative follows from its existence, but the proof will not be given here.