CHAPTER XVII

SPECIAL INFINITE DEVELOPMENTS

171. The trigonometric functions. If m is an odd integer, say $m = 2n + 1$, De Moivre's Theorem (§ 72) gives

$$\frac{\sin m\phi}{m \sin \phi} = \cos^{2n}\phi - \frac{(m-1)(m-2)}{3!}\cos^{2n-2}\phi \sin^2\phi + \cdots, \qquad (1)$$

where by virtue of the relation $\cos^2\phi = 1 - \sin^2\phi$ the right-hand member is a polynomial of degree n in $\sin^2\phi$. From the left-hand side it is seen that the value of the polynomial is 1 when $\sin\phi = 0$ and that the n roots of the polynomials are

$$\sin^2\pi/m, \qquad \sin^2 2\pi/m, \qquad \cdots, \qquad \sin^2 n\pi/m.$$

Hence the polynomial may be factored in the form

$$\frac{\sin m\phi}{m \sin \phi} = \left(1 - \frac{\sin^2\phi}{\sin^2\pi/m}\right)\left(1 - \frac{\sin^2\phi}{\sin^2 2\pi/m}\right)\cdots\left(1 - \frac{\sin^2\phi}{\sin^2 n\pi/m}\right). \qquad (2)$$

If the substitutions $\phi = x/m$ and $\phi = ix/m$ be made,

$$\frac{\sin x}{m \sin x/m} = \left(1 - \frac{\sin^2 x/m}{\sin^2\pi/m}\right)\left(1 - \frac{\sin^2 x/m}{\sin^2 2\pi/m}\right)\cdots\left(1 - \frac{\sin^2 x/m}{\sin^2 n\pi/m}\right), \qquad (3)$$

$$\frac{\sinh x}{m \sinh x/m} = \left(1 + \frac{\sinh^2 x/m}{\sin^2\pi/m}\right)\left(1 + \frac{\sinh^2 x/m}{\sin^2 2\pi/m}\right)\cdots\left(1 + \frac{\sinh^2 x/m}{\sin^2 n\pi/m}\right). \qquad (3')$$

Now if m be allowed to become infinite, passing through successive odd integers, these equations remain true and it would appear that the limiting relations would hold:

$$\frac{\sin x}{x} = \left(1 - \frac{x^2}{\pi^2}\right)\left(1 - \frac{x^2}{2^2\pi^2}\right)\cdots = \prod_1^\infty\left(1 - \frac{x^2}{k^2\pi^2}\right), \qquad (4)$$

$$\frac{\sinh x}{x} = \left(1 + \frac{x^2}{\pi^2}\right)\left(1 + \frac{x^2}{2^2\pi^2}\right)\cdots = \prod_1^\infty\left(1 + \frac{x^2}{k^2\pi^2}\right), \qquad (4')$$

since $$\lim_{m=\infty}\frac{\sin^2\dfrac{x}{m}}{\sin^2\dfrac{k\pi}{m}} = \lim_{m=\infty}\frac{\left(\dfrac{x}{m} - \dfrac{1}{6}\dfrac{x^3}{m^3} + \cdots\right)^2}{\left(\dfrac{k\pi}{m} - \dfrac{1}{6}\left(\dfrac{k\pi}{m}\right)^3 + \cdots\right)^2} = \frac{x^2}{k^2\pi^2}.$$