CHAPTER XV

THE CALCULUS OF VARIATIONS

155. The treatment of the simplest case. The integral

$$I = {}_C\int_A^B F(x, y, y')\,dx = {}_C\int_A^B \Phi(x, y, dx, dy), \tag{1}$$

where Φ is homogeneous of the first degree in dx and dy, may be evaluated along any curve C between the limits A and B by reduction to an ordinary integral. For if C is given by $y = f(x)$,

$$I = {}_C\int_A^B F(x, y, y')\,dx = \int_{x_0}^{x_1} F(x, f(x), f'(x))\,dx;$$

and if C is given by $x = \phi(t)$, $y = \psi(t)$,

$$I = {}_C\int_A^B \Phi(x, y, dx, dy) = \int_{t_0}^{t_1} \Phi(\phi, \psi, \phi', \psi')\,dt.$$

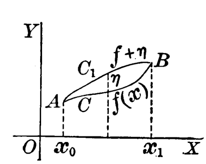
The ordinary line integral (§ 122) is merely the special case in which $\Phi = Pdx + Qdy$ and $F = P + Qy'$. In general the value of I will depend on the path C of integration; *the problem of the calculus of variations is to find that path which will make I a maximum or minimum relative to neighboring paths.*

If a second path C_1 be $y = f(x) + \eta(x)$, where $\eta(x)$ is a small quantity which vanishes at x_0 and x_1, a whole family of paths is given by

$$y = f(x) + \alpha\eta(x), \qquad -1 \leqq \alpha \leqq 1, \qquad \eta(x_0) = \eta(x_1) = 0,$$

and the value of the integral

$$I(\alpha) = \int_{x_0}^{x_1} F(x, f + \alpha\eta, f' + \alpha\eta')\,dx, \tag{1'}$$



taken along the different paths of the family, becomes a function of α; in particular $I(0)$ and $I(1)$ are the values along C and C_1. Under appropriate assumptions as to the continuity of F and its partial derivatives F'_x, F'_y, $F'_{y'}$, the function $I(\alpha)$ will be continuous and have a continuous derivative which may be found by differentiating under the sign (§ 119); then

$$I'(\alpha) = \int_{x_0}^{x_1} [\eta F'_y(x, f + \alpha\eta, f' + \alpha\eta') + \eta' F'_{y'}(x, f + \alpha\eta, f' + \alpha\eta')]\,dx.$$