CHAPTER XIV

SPECIAL FUNCTIONS DEFINED BY INTEGRALS

147. The Gamma and Beta functions. The two integrals

$$\Gamma(n) = \int_0^\infty x^{n-1}e^{-x}dx, \qquad \mathrm{B}(m, n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx \tag{1}$$

converge when $n > 0$ and $m > 0$, and hence define functions of the parameters n or n and m for all positive values, zero not included. Other forms may be obtained by changes of variable. Thus

$$\Gamma(n) = 2\int_0^\infty y^{2n-1}e^{-y^2}dy, \qquad \text{by} \quad x = y^2, \tag{2}$$

$$\Gamma(n) = \int_0^1 \left(\log\frac{1}{y}\right)^{n-1} dy, \qquad \text{by} \quad e^{-x} = y, \tag{3}$$

$$\mathrm{B}(m, n) = \int_0^1 y^{n-1}(1-y)^{m-1}dy = \mathrm{B}(n, m), \quad \text{by} \quad x = 1-y, \tag{4}$$

$$\mathrm{B}(m, n) = \int_0^\infty \frac{y^{m-1}dy}{(1+y)^{m+n}}, \qquad \text{by} \quad x = \frac{y}{1+y}, \tag{5}$$

$$\mathrm{B}(m, n) = 2\int_0^{\frac{\pi}{2}} \sin^{2m-1}\phi\cos^{2n-1}\phi d\phi, \qquad \text{by} \quad x = \sin^2\phi. \tag{6}$$

If the original form of $\Gamma(n)$ be integrated by parts, then

$$\Gamma(n) = \int_0^\infty x^{n-1}e^{-x}dx = \frac{1}{n}x^ne^{-x}\Big]_0^\infty + \frac{1}{n}\int_0^\infty x^ne^{-x}dx = \frac{1}{n}\Gamma(n+1).$$

The resulting relation $\Gamma(n+1) = n\Gamma(n)$ shows that the values of the Γ-function for $n+1$ may be obtained from those for n, and that consequently the values of the function will all be determined if the values over a unit interval are known. Furthermore

$$\begin{aligned}\Gamma(n+1) &= n\Gamma(n) = n(n-1)\Gamma(n-1)\\ &= n(n-1)\cdots(n-k)\Gamma(n-k)\end{aligned} \tag{7}$$

is found by successive reduction, where k is any integer less than n. If in particular n is an integer and $k = n-1$, then

$$\Gamma(n+1) = n(n-1)\cdots 2\cdot 1\cdot\Gamma(1) = n!\,\Gamma(1) = n!; \tag{8}$$