CHAPTER XIII

ON INFINITE INTEGRALS

140. Convergence and divergence. The definite integral, and hence for theoretical purposes the indefinite integral, has been defined,

$$\int_a^b f(x)\,dx, \qquad F(x) = \int_a^x f(x)\,dx,$$

when the function $f(x)$ is *limited* in the interval a to b, or a to x; the proofs of various propositions have depended essentially on the fact that *the integrand remained finite over the finite interval of integration* (§§ 16–17, 28–30). Nevertheless problems which call for the determination of the area between a curve and its asymptote, say the area under the witch or cissoid,

$$\int_{-\infty}^{+\infty} \frac{8\,a^3dx}{x^2+4\,a^2} = 4\,a^2 \tan^{-1}\frac{x}{2\,a}\Big|_{-\infty}^{+\infty} = 4\,\pi a^2, \qquad 2\int_0^{2a} \frac{x^{\frac{3}{2}}dx}{\sqrt{2\,a-x}} = 3\,\pi a^2,$$

have arisen and have been treated as a matter of course.* The integrals of this sort require some special attention.

When the integrand of a definite integral becomes infinite within or at the extremities of the interval of integration, or when one or both of the limits of integration become infinite, the integral is called an infinite integral and is defined, not as the limit of a sum, but as the limit of an integral with a variable limit, that is, as the limit of a function. Thus

$$\int_a^\infty f(x)\,dx = \lim_{x=\infty}\left[F(x) = \int_a^x f(x)dx\right], \qquad \text{infinite upper limit,}$$

$$\int_a^b f(x)\,dx = \lim_{x\doteq b}\left[F(x) = \int_a^x f(x)\,dx\right], \qquad \text{integrand } f(b) = \infty.$$

These definitions may be illustrated by figures which show the connection with the idea of area between a curve and its asymptote. Similar definitions would be given if the lower limit were $-\infty$ or if the integrand became infinite at $x = a$. If the integrand were infinite at some intermediate point of the interval, the interval would be subdivided into two intervals and the definition would be applied to each part.

* Here and below the construction of figures is left to the reader.