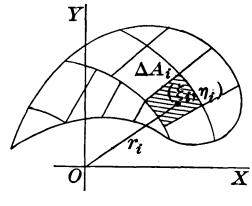
CHAPTER XII

ON MULTIPLE INTEGRALS

129. Double sums and double integrals. Suppose that a body of matter is so thin and flat that it can be considered to lie in a plane. If any small portion of the body surrounding a given point $P(x, y)$ be considered, and if its mass be denoted by Δm and its area by ΔA, the average (surface) density of the portion is the quotient $\Delta m/\Delta A$, and the actual density at the point P is defined as the limit of this quotient when $\Delta A \doteq 0$, that is,

$$D(x, y) = \lim_{\Delta A \doteq 0} \frac{\Delta m}{\Delta A}.$$

The density may vary from point to point. Now conversely suppose that the density $D(x, y)$ of the body is a known function of (x, y) and that it be required to find the total mass of the body. Let the body be considered as divided up into a large number of pieces each of which is *small in every direction*, and let ΔA_i be the area of any piece. If (ξ_i, η_i) be any point in ΔA_i, the density at that point is $D(\xi_i, \eta_i)$ and the amount of matter in the piece is approximately $D(\xi_i, \eta_i)\Delta A_i$ provided the density be regarded as continuous, that is, as not varying much over so small an area. Then the sum



$$D(\xi_1, \eta_1)\Delta A_1 + D(\xi_2, \eta_2)\Delta A_2 + \cdots + D(\xi_n, \eta_n)\Delta A_n = \sum D(\xi_i, \eta_i)\Delta A_i,$$

extended over all the pieces, is an approximation to the total mass, and may be sufficient for practical purposes if the pieces be taken tolerably small.

The process of dividing a body up into a large number of small pieces of which it is regarded as the sum is a device often resorted to; for the properties of the small pieces may be known approximately, so that the corresponding property for the whole body can be obtained approximately by summation. Thus by definition the moment of inertia of a small particle of matter relative to an axis is mr^2, where m is the mass of the particle and r its distance from the axis. If therefore the moment of inertia of a plane body with respect to an axis perpendicular