PART III. INTEGRAL CALCULUS

CHAPTER XI

ON SIMPLE INTEGRALS

118. Integrals containing a parameter. Consider

$$\phi(\alpha) = \int_{x_0}^{x_1} f(x, \alpha)\, dx, \qquad (1)$$

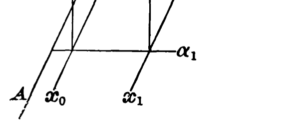
a definite integral which contains in the integrand a parameter α. If the indefinite integral is known, as in the case

$$\int \cos \alpha x dx = \frac{1}{\alpha} \sin \alpha x, \qquad \int_0^{\frac{\pi}{2}} \cos \alpha x dx = \frac{1}{\alpha} \sin \alpha x \Big|_0^{\frac{\pi}{2}} = \frac{1}{\alpha},$$

it is seen that the indefinite integral is a function of x and α, and that the definite integral is a function of α alone because the variable x disappears on the substitution of the limits. If the limits themselves depend on α, as in the case

$$\int_{\frac{1}{\alpha}}^{\alpha} \cos \alpha x dx = \frac{1}{\alpha} \sin \alpha x \Big|_{\frac{1}{\alpha}}^{\alpha} = \frac{1}{\alpha} (\sin \alpha^2 - \sin 1),$$

the integral is still a function of α.



In many instances the indefinite integral in (1) cannot be found explicitly and it then becomes necessary to discuss the continuity, differentiation, and integration of the function $\phi(\alpha)$ defined by the integral without having recourse to the actual evaluation of the integral; in fact these discussions may be required in order to effect that evaluation. Let the limits x_0 and x_1 be taken as constants independent of α. Consider the range of values $x_0 \leqq x \leqq x_1$ for x, and let $\alpha_0 \leqq \alpha \leqq \alpha_1$ be the range of values over which the function $\phi(\alpha)$ is to be discussed. The function $f(x, \alpha)$ may be plotted as the surface $z = f(x, \alpha)$ over the rectangle of values for (x, α). The