CHAPTER X

DIFFERENTIAL EQUATIONS IN MORE THAN TWO VARIABLES

109. Total differential equations. An equation of the form

$$P(x, y, z)\,dx + Q(x, y, z)\,dy + R(x, y, z)\,dz = 0, \tag{1}$$

involving the differentials of three variables is called a *total differential equation.* A similar equation in any number of variables would also be called total; but the discussion here will be restricted to the case of three. If definite values be assigned to x, y, z, say a, b, c, the equation becomes

$$A\,dx + B\,dy + C\,dz = A(x-a) + B(y-b) + C(z-c) = 0, \tag{2}$$

where x, y, z are supposed to be restricted to values near a, b, c, and represents a small portion of a plane passing through (a, b, c). From the analogy to the lineal element (§ 85), such a portion of a plane may be called a *planar element.* The differential equation therefore represents an infinite number of planar elements, one passing through each point of space.

Now any family of surfaces $F(x, y, z) = C$ also represents an infinity of planar elements, namely, the portions of the tangent planes at every point of all the surfaces in the neighborhood of their respective points of tangency. In fact

$$dF = F'_x dx + F'_y dy + F'_z dz = 0 \tag{3}$$

is an equation similar to (1). If the planar elements represented by (1) and (3) are to be the same, the equations cannot differ by more than a factor $\mu(x, y, z)$. Hence

$$F'_x = \mu P, \qquad F'_y = \mu Q, \qquad F'_z = \mu R.$$

If a function $F(x, y, z) = C$ can be found which satisfies these conditions, it is said to be the integral of (1), and the factor $\mu(x, y, z)$ by which the equations (1) and (3) differ is called an *integrating factor* of (1). Compare § 91.

It may happen that $\mu = 1$ and that (1) is thus an *exact* differential. In this case the conditions

$$P'_y = Q'_x, \qquad Q'_z = R'_y, \qquad R'_x = P'_z, \tag{4}$$