CHAPTER VIII

THE COMMONER ORDINARY DIFFERENTIAL EQUATIONS

89. Integration by separating the variables. If a differential equation of the first order may be solved for y' so that

$$y' = \phi(x, y) \quad \text{or} \quad M(x, y)\,dx + N(x, y)\,dy = 0 \tag{1}$$

(where the functions ϕ, M, N are single valued or where only one specific branch of each function is selected in case the solution leads to multiple valued functions), the differential equation involves only the first power of the derivative and is said to be of the first degree. If, furthermore, it so happens that the functions ϕ, M, N are products of functions of x and functions of y so that the equation (1) takes the form

$$y' = \phi_1(x)\,\phi_2(y) \quad \text{or} \quad M_1(x)\,M_2(y)\,dx + N_1(x)\,N_2(y)\,dy = 0, \tag{2}$$

it is clear that the variables may be separated in the manner

$$\frac{dy}{\phi_2(y)} = \phi_1(x)\,dx \quad \text{or} \quad \frac{M_1(x)}{N_1(x)}\,dx + \frac{N_2(y)}{M_2(y)}\,dy = 0, \tag{2'}$$

and the integration is then immediately performed by integrating each side of the equation. It was in this way that the numerous problems considered in Chap. VII were solved.

As an example consider the equation $yy' + xy^2 = x$. Here

$$ydy + x(y^2 - 1)\,dx = 0 \quad \text{or} \quad \frac{ydy}{y^2 - 1} + xdx = 0,$$

and $\qquad \frac{1}{2}\log(y^2 - 1) + \frac{1}{2}x^2 = C \quad \text{or} \quad (y^2 - 1)\,e^{x^2} = C.$

The second form of the solution is found by taking the exponential of both sides of the first form after multiplying by 2.

In some differential equations (1) in which the variables are not immediately separable as above, the introduction of some change of variable, whether of the dependent or independent variable or both, may lead to a differential equation in which the new variables are separated and the integration may be accomplished. The selection of the proper change of variable is in general a matter for the exercise of ingenuity; succeeding paragraphs, however, will point out some special