PART II. DIFFERENTIAL EQUATIONS

CHAPTER VII

GENERAL INTRODUCTION TO DIFFERENTIAL EQUATIONS

81. Some geometric problems. The application of the differential calculus to plane curves has given a means of determining some geometric properties of the curves. For instance, the length of the subnormal of a curve (§ 7) is ydy/dx, which in the case of the parabola $y^2 = 4px$ is $2p$, that is, the subnormal is constant. Suppose now it were desired conversely to find all curves for which the subnormal is a given constant m. The statement of this problem is evidently contained in the equation

$$y\frac{dy}{dx} = m \quad \text{or} \quad yy' = m \quad \text{or} \quad ydy = mdx.$$

Again, the radius of curvature of the lemniscate $r^2 = a^2 \cos 2\phi$ is found to be $R = a^2/3r$, that is, the radius of curvature varies inversely as the radius. If conversely it were desired to find all curves for which the radius of curvature varies inversely as the radius of the curve, the statement of the problem would be the equation

$$\frac{\left[r^2 + \left(\frac{dr}{d\phi}\right)^2\right]^{\frac{3}{2}}}{r^2 - r\frac{d^2r}{d\phi^2} + 2\left(\frac{dr}{d\phi}\right)^2} = \frac{k}{r},$$

where k is a constant called a factor of proportionality.*

Equations like these are unlike ordinary algebraic equations because, in addition to the variables x, y or r, ϕ and certain constants m or k, they contain also derivatives, as dy/dx or $dr/d\phi$ and $d^2r/d\phi^2$, of one of the variables with respect to the other. An equation which contains

* Many problems in geometry, mechanics, and physics are stated in terms of variation. For purposes of analysis the statement x varies as y, or $x \propto y$, is written as $x = ky$, introducing a constant k called a factor of proportionality to convert the variation into an equation. In like manner the statement x varies inversely as y, or $x \propto 1/y$, becomes $x = k/y$, and x varies jointly with y and z becomes $x = kyz$.