CHAPTER VI

COMPLEX NUMBERS AND VECTORS

70. Operators and operations. If an entity u is changed into an entity v by some law, the change may be regarded as an *operation* performed upon u, the *operand*, to convert it into v; and if f be introduced as the symbol of the operation, the result may be written as $v = fu$. For brevity the symbol f is often called an *operator*. Various sorts of operand, operator, and result are familiar. Thus if u is a positive number n, the application of the operator $\sqrt{}$ gives the square root; if u represents a range of values of a variable x, the expression $f(x)$ or fx denotes a function of x; if u be a function of x, the operation of differentiation may be symbolized by D and the result Du is the derivative; the symbol of definite integration $\int_a^b (*)\,d*$ converts a function $u(x)$ into a number; and so on in great variety.

The reason for making a short study of operators is that a considerable number of the concepts and rules of arithmetic and algebra may be so defined for operators themselves as to lead to a *calculus of operations* which is of frequent use in mathematics; the single application to the integration of certain differential equations (§ 95) is in itself highly valuable. The fundamental concept is that of a *product: If u is operated upon by f to give $fu = v$ and if v is operated upon by g to give $gv = w$, so that*

$$fu = v, \qquad gv = gfu = w, \qquad gfu = w, \tag{1}$$

then the operation indicated as gf which converts u directly into w is called the product of f by g. If the functional symbols sin and log be regarded as operators, the symbol log sin could be regarded as the product. The transformations of turning the xy-plane over on the x-axis, so that $x' = x$, $y' = -y$, and over the y-axis, so that $x' = -x$, $y' = y$, may be regarded as operations; the combination of these operations gives the transformation $x' = -x$, $y' = -y$, which is equivalent to rotating the plane through 180° about the origin.

The products of arithmetic and algebra satisfy the *commutative law* $gf = fg$, that is, the products of g by f and of f by g are equal. This is not true of operators in general, as may be seen from the fact that