CHAPTER V

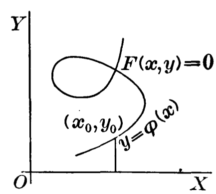
PARTIAL DIFFERENTIATION; IMPLICIT FUNCTIONS

56. The simplest case; $F(x, y) = 0$. The total differential

$$dF = F'_x dx + F'_y dy = d0 = 0$$

indicates
$$\frac{dy}{dx} = -\frac{F'_x}{F'_y}, \qquad \frac{dx}{dy} = -\frac{F'_y}{F'_x} \tag{1}$$

as the derivative of y by x, or of x by y, where y is defined as a function of x, or x as a function of y, by the relation $F(x, y) = 0$; and this method of obtaining a derivative of an *implicit function* without solving explicitly for the function has probably been familiar long before the notion of a partial derivative was obtained. The relation $F(x, y) = 0$ is pictured as a curve, and the function $y = \phi(x)$, which would be obtained by solution, is considered as multiple valued or as restricted to some definite portion or branch of the curve $F(x, y) = 0$. If the results (1) are to be applied to find the derivative at some point (x_0, y_0) of the curve $F(x, y) = 0$, it is necessary that at that point the denominator F'_y or F'_x should not vanish.



These pictorial and somewhat vague notions may be stated precisely as a *theorem* susceptible of proof, namely: Let x_0 be any real value of x such that 1°, the equation $F(x_0, y) = 0$ has a real solution y_0; and 2°, the function $F(x, y)$ regarded as a function of two independent variables (x, y) is continuous and has continuous first partial derivatives F'_x, F'_y in the neighborhood of (x_0, y_0); and 3°, the derivative $F'_y(x_0, y_0) \neq 0$ does not vanish for (x_0, y_0); then $F(x, y) = 0$ may be solved (theoretically) as $y = \phi(x)$ in the vicinity of $x = x_0$ and in such a manner that $y_0 = \phi(x_0)$, that $\phi(x)$ is continuous in x, and that $\phi(x)$ has a derivative $\phi'(x) = -F'_x/F'_y$; and the solution is unique. This is the fundamental theorem on implicit functions for the simple case, and the proof follows.

By the conditions on F'_x, F'_y, the Theorem of the Mean is applicable. Hence

$$F(x, y) - F(x_0, y_0) = F(x, y) = (hF'_x + kF'_y)_{x_0 + \theta h,\, y_0 + \theta k}. \tag{2}$$

Furthermore, in any square $|h| < \delta$, $|k| < \delta$ surrounding (x_0, y_0) and sufficiently small, the continuity of F'_x insures $|F'_x| < M$ and the continuity of F'_y taken with