NOTE ON THE THEORY OF IDEALS

The following is a brief explanation of the theory of ideals of algebraic numbers* and functions and the relation in which the theory given in the preceding pages stands with respect to it.

Gauss (*Disquisitiones Arithmeticae* (1801)) was the first to consider the laws of factorisation in a domain of whole numbers other than that of rational whole numbers $0, \pm 1, \pm 2, \ldots$. He proved that two given complex whole numbers $a+b\sqrt{-1}$, $c+d\sqrt{-1}$ (a, b, c, d rational integers) have always an H.C.F. and that any such number is a unique product of prime factors. Kummer (*J. reine angew. Math.* 35 (1847), 40 (1850), 53 (1857)) in extending the research to a larger class of whole numbers found that these properties were no longer absolutely true. Nevertheless he succeeded in making such numbers amenable to all the simpler laws of rational integers by introducing certain *ideal numbers* not existing in the domain considered; and thus laid the foundation of the theory of factorisation of whole algebraic numbers. Finally Dedekind (D), by using *ideals* instead of ideal numbers, extended the theory to the whole numbers of any algebraic corpus and to whole algebraic functions of one variable (DW); while Kronecker (Kr) extended the same theory of factorisation to algebraic functions in general. Kronecker went still further; he gave the first steps of a *general* theory of ideals of algebraic functions (Kr, p. 77) under the name of *modular systems*. In this general theory factorisation plays only a subsidiary part, since an ideal which is not prime is not in general a product of prime ideals.

Modules of whole rational functions (as defined pp. 1, 2 above) are ideals and modules in the sense of Dedekind; and the theory of such modules is the necessary starting point of the general theory of ideals.

An *algebraic number* is any root α of an algebraic equation

$$a_0x^m + a_1x^{m-1} + \ldots + a_m = 0$$

* The following are notable general accounts of the theory of algebraic numbers:

D. Hilbert. "Bericht über die Theorie der algebraischen Zahlkörper" (*Jahresb. d. deutschen Math.-Verein.*, Berlin (1897), Bd. IV).

H. Weber. *Lehrbuch der Algebra* (Brunswick, 2nd ed. (1899), Bd. II, p. 553).

G. B. Mathews. "Number" (*Ency. Brit.*, Cambridge, 11th ed. (1911), Vol. 19, p. 847).

For other references to the arithmetic theory of algebraic numbers and functions see (D), (DW), (K), and (Kr), p. xiii.