III. GENERAL PROPERTIES OF MODULES

23. Several arithmetical terms are used in connection with modules suggesting an analogy between the properties of polynomials and the properties of natural numbers. Two modules have a G.C.M., an L.C.M., a product, and a residual (integral quotient); but no sum or difference. Also a prime module answers to a prime number and a primary module to a power of a prime number. Such terms must not be used for making deductions by analogy.

Definitions. Any member F of a module M is said to *contain* M. Also the module (F) contains M. It is immaterial in this statement as in many others whether we regard F as a polynomial or a module. The term *contains* is used as an extension and generalisation of the phrase *is divisible by*.

More generally a module M is said to *contain* another M' if every member of M contains M'; and this will be the case if every member of the basis of M contains M'. Thus $(F_1, F_2, \ldots, F_k)$ contains $(F_1, F_2, \ldots, F_{k+1})$, and a module becomes less by adding new members to it.

If M contains M' and M' contains M we say that M, M' are the same module, or $M = M'$.

If M contains M' the spread of M contains the spread of M', but the converse is not true in general.

If in a given finite or infinite set of modules there is one which is contained in every other one, that one is called the *least* module of the set; or if there is one which contains every other one, that one is called the *greatest* module of the set. Two modules cannot be compared as to greater or less unless one contains the other.

There is a module which is contained in all modules, the *unit module* (1). Also (0) may be conceived of as a module which contains all modules; but it seldom comes into consideration and will not be mentioned again. These two modules are called non-proper modules, and all others are *proper* modules. In general by a module a proper module is to be understood.

The G.C.M. of k given modules $M_1, M_2, \ldots, M_k$ is the greatest of all modules M contained in M_1 and $M_2 \ldots$ and M_k, and is denoted by $(M_1, M_2, \ldots, M_k)$. In order that M may be contained in each of $M_1, M_2, \ldots, M_k$, or that each of $M_1, M_2, \ldots, M_k$ may contain M, it is