Appendix A
The Maximum Principles

In this chapter we summarize the maximum principles that will be used in this book. For more general theory, see Friedman [17], Protter and Weinberger [39] and Gilbarg and Trudinger [18].

The following is the standing assumptions in this chapter. Let n be an integer with $n \geq 1$. Let E be an open set in $\mathbb{R}^{n+1}$ with

$$\begin{aligned} \partial E \cap \{(x,0) \,|\, x \in \mathbb{R}^n\} &\neq \emptyset, \\ E \cap (\mathbb{R}^n \times (-\infty, 0]) &= \emptyset. \end{aligned}$$

For given $T > 0$, we define

$$\begin{aligned} \Gamma_T &= \{(x,t) \in \partial E \,|\, t \leq T\}, \\ E_T &= \{(x,t) \in E \,|\, t \leq T\}. \end{aligned}$$

If $E = \{(x,t) \,|\, x \in \mathbb{R}^n, t > 0\}$, we have $E_T = \mathbb{R}^n \times (0,T]$ and $\Gamma_T = \{(x,0) \,|\, x \in \mathbb{R}^n\}$.

For any point P in E_T, we denote by $S(P)$ the set of all points Q in E_T that can be connected to P by a simple continuous curve in E_T along which the t-coordinate is nondecreasing from Q to P. See Figure A.1.

We always assume that $u(x,t)$ and $\boldsymbol{u}(x,t) = (u_1(x,t), \ldots, u_m(x,t))$ are continuous in $E_T \cup \Gamma_T$ and they are twice continuously differentiable in x and continuously differentiable in t in E_T.

A.1 Maximum Principles for Differential Inequalities

Let $a_{ij}(x,t)$, $b(x,t) = (b_j(x,t))_{1 \leq j \leq n}$ and $h(x,t)$ be continuous functions with

$$\sup_{(x,t) \in E_T} |a_{ij}(x,t)| < \infty, \qquad \sup_{(x,t) \in E_T} \frac{|b_j(x,t)|}{1+|x|} < \infty, \qquad \sup_{(x,t) \in E_T} \frac{|h(x,t)|}{1+|x|^2} < \infty$$

for all $1 \leq i,j \leq n$. Now we assume

$$a_{ji}(x,t) = a_{ij}(x,t) \qquad \text{in } E_T$$

for all $1 \leq i,j \leq n$. We assume that there exists $\mu_0 > 0$ such that we have

$$\sum_{i,j=1}^{n} a_{ij}(x,t)\xi_i\xi_j \geq \mu_0 |\xi|^2$$

every $(x,t) \in E_T$ and every $\xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n$.