CHAPTER 11

Shift operators

In this chapter we study an integer shift of spectral parameters $\lambda_{j,\nu}$ of the Fuchsian equation $P_{\mathbf{m}}(\lambda)u = 0$. Here $P_{\mathbf{m}}(\lambda)$ is the universal operator (cf. Theorem 6.14) corresponding to the spectral type $\mathbf{m} = \big(m_{j,\nu}\big)_{\substack{j=0,\dots,p\\ \nu=1,\dots,n_j}}$. For simplicity, we assume that $\mathbf{m}$ is rigid in this chapter unless otherwise stated.

11.1. Construction of shift operators and contiguity relations

First we construct shift operators for general shifts.

Definition 11.1. Fix a tuple of partitions $\mathbf{m} = \big(m_{j,\nu}\big)_{\substack{j=0,\dots,p\\ \nu=1,\dots,n_j}} \in \mathcal{P}_{p+1}^{(n)}$. Then a set of integers $\big(\epsilon_{j,\nu}\big)_{\substack{j=0,\dots,p\\ \nu=1,\dots,n_j}}$ parametrized by j and ν is called a *shift compatible to* $\mathbf{m}$ if

$$\sum_{j=0}^{p}\sum_{\nu=1}^{n_j}\epsilon_{j,\nu}m_{j,\nu} = 0. \tag{11.1}$$

Theorem 11.2 (shift operator). *Fix a shift $(\epsilon_{j,\nu})$ compatible to $\mathbf{m} \in \mathcal{P}_{p+1}^{(n)}$. Then there is a shift operator $R_{\mathbf{m}}(\epsilon,\lambda) \in W[x]\otimes\mathbb{C}[\lambda_{j,\nu}]$ which gives a homomorphism of the equation $P_{\mathbf{m}}(\lambda')v = 0$ to $P_{\mathbf{m}}(\lambda)u = 0$ defined by $v = R_{\mathbf{m}}(\epsilon,\lambda)u$. Here the Riemann scheme of $P_{\mathbf{m}}(\lambda)$ is $\{\lambda_{\mathbf{m}}\} = \big\{[\lambda_{j,\nu}]_{(m_{j,\nu})}\big\}_{\substack{j=0,\dots,p\\ \nu=1,\dots,n_j}}$ and that of $P_{\mathbf{m}}(\lambda')$ is $\{\lambda'_{\mathbf{m}}\}$ defined by $\lambda'_{j,\nu} = \lambda_{j,\nu} + \epsilon_{j,\nu}$. Moreover we may assume* $\operatorname{ord} R_{\mathbf{m}}(\epsilon,\lambda) < \operatorname{ord}\mathbf{m}$ *and $R_{\mathbf{m}}(\epsilon,\lambda)$ never vanishes as a function of λ and then $R_{\mathbf{m}}(\epsilon,\lambda)$ is uniquely determined up to a constant multiple.*

Putting

$$\tau = \big(\tau_{j,\nu}\big)_{\substack{0\le j\le p\\ 1\le\nu\le n_j}} \quad \text{with} \quad \tau_{j,\nu} := \big(2 + (p-1)n\big)\delta_{j,0} - m_{j,\nu} \tag{11.2}$$

and $d = \operatorname{ord} R_{\mathbf{m}}(\epsilon,\lambda)$, we have

$$P_{\mathbf{m}}(\lambda+\epsilon)R_{\mathbf{m}}(\epsilon,\lambda) = (-1)^d R_{\mathbf{m}}(\epsilon,\tau-\lambda-\epsilon)^* P_{\mathbf{m}}(\lambda) \tag{11.3}$$

under the notation in Theorem 4.19 ii).

Proof. We will prove the theorem by the induction on $\operatorname{ord}\mathbf{m}$. The theorem is clear if $\operatorname{ord}\mathbf{m} = 1$.

We may assume that $\mathbf{m}$ is monotone. Then the reduction $\{\tilde\lambda_{\tilde{\mathbf{m}}}\}$ of the Riemann scheme is defined by (10.33). Hence putting

$$\begin{cases} \tilde\epsilon_1 = \epsilon_{0,1} + \cdots + \epsilon_{p,1},\\ \tilde\epsilon_{j,\nu} = \epsilon_{j,\nu} + \big((-1)^{\delta_{j,0}} - \delta_{\nu,1}\big)\tilde\epsilon_1 \quad (j = 0,\dots,p,\ \nu = 1,\dots,n_j), \end{cases} \tag{11.4}$$