CHAPTER 7

A Kac-Moody root system

In this chapter we explain a correspondence between spectral types and roots of a Kac-Moody root system. The correspondence was first introduced by Crawley-Boevey [**CB**]. In §7.2 we study fundamental tuples through this correspondence.

7.1. Correspondence with a Kac-Moody root system

We review a Kac-Moody root system to describe the combinatorial structure of middle convolutions on the spectral types. Its relation to Deligne-Simpson problem is first clarified by [**CB**].

Let

$$I := \{0,\ (j,\nu)\,;\ j = 0,1,\ldots,\ \nu = 1,2,\ldots\}. \tag{7.1}$$

be a set of indices and let $\mathfrak{h}$ be an infinite dimensional real vector space with the set of basis Π, where

$$\Pi = \{\alpha_i\,;\ i \in I\} = \{\alpha_0,\ \alpha_{j,\nu}\,;\ j = 0,1,2,\ldots,\ \nu = 1,2,\ldots\}. \tag{7.2}$$

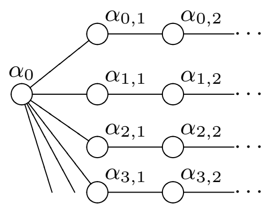
Put

$$I' := I \setminus \{0\}, \qquad \Pi' := \Pi \setminus \{\alpha_0\}, \tag{7.3}$$

$$Q := \sum_{\alpha\in\Pi} \mathbb{Z}\alpha \ \supset\ Q_+ := \sum_{\alpha\in\Pi} \mathbb{Z}_{\geq 0}\alpha. \tag{7.4}$$

We define an indefinite symmetric bilinear form on $\mathfrak{h}$ by

$$\begin{aligned} (\alpha|\alpha) &= 2 \qquad (\alpha \in \Pi),\\ (\alpha_0|\alpha_{j,\nu}) &= -\delta_{\nu,1},\\ (\alpha_{i,\mu}|\alpha_{j,\nu}) &= \begin{cases} 0 & (i \neq j \ \text{ or }\ |\mu-\nu| > 1),\\ -1 & (i = j \ \text{ and }\ |\mu-\nu| = 1). \end{cases} \end{aligned} \tag{7.5}$$



The element of Π is called the *simple root* of a Kac-Moody root system and the *Weyl group* W_∞ of this Kac-Moody root system is generated by the *simple reflections* s_i with $i \in I$. Here the *reflection* with respect to an element $\alpha \in \mathfrak{h}$ satisfying $(\alpha|\alpha) \neq 0$ is the linear transformation

$$s_\alpha\ :\ \mathfrak{h} \ni x \mapsto x - 2\frac{(x|\alpha)}{(\alpha|\alpha)}\alpha \in \mathfrak{h} \tag{7.6}$$

and

$$s_i = s_{\alpha_i} \ \text{ for }\ i \in I. \tag{7.7}$$

In particular $s_i(x) = x - (\alpha_i|x)\alpha_i$ for $i \in I$ and the subgroup of W_∞ generated by s_i for $i \in I \setminus \{0\}$ is denoted by W'_∞.

The Kac-Moody root system is determined by the set of simple roots Π and its Weyl group W_∞ and it is denoted by (Π, W_∞).

Denoting $\sigma(\alpha_0) = \alpha_0$ and $\sigma(\alpha_{j,\nu}) = \alpha_{\sigma(j),\nu}$ for $\sigma \in \mathfrak{S}_\infty$, we put

$$\widetilde{W}_\infty := \mathfrak{S}_\infty \ltimes W_\infty, \tag{7.8}$$