Chapter 4

Topology of orbifolds

This section begins by reviewing the theory of the compact group actions on manifolds. Then we move on to define orbifold and their maps. We also cover the groupoid definition. We discuss the differentiable structures on orbifolds and the triangulation of orbifolds following the book [Verona (1984)]. We expose the covering theory using the fiber-product approach following Thurston and the path-approach following Haefliger. We make some computations of the fundamental groups. Finally, we relate the fundamental groups with the covering spaces.

We tried to make the abstract definitions into more concrete forms here; however, in many respect, the abstract definitions give us a more accurate sense of what an orbifold is. (For examples, see the article [Lerman (2010)].) This section is somewhat technical but essential to the developments later.

4.1 Compact group actions

Although we need only the result for finite group actions, we will study the situations when G is a compact Lie group. Let X be a space. We are given a group action $G \times X \to X$ with $e(x) = x$ for all x and $gh(x) = g(h(x))$. That is, we have a homomorphism $G \to \mathbf{Diff}(X)$ so that the product operation corresponds to the composition. In this case, X with the action is said to be a G-space.

An *equivariant* map $\phi : X \to Y$ between G-spaces is a map so that $\phi(g(x)) = g(\phi(x))$ for all $x \in X$. An *isotropy subgroup* G_x is defined as $\{g \in G | g(x) = x\}$. We note that $G_{g(x)} = gG_xg^{-1}$ and $G_x \subset G_{\phi(x)}$ for an equivariant map ϕ.

Theorem 4.1.1 (Tietze-Gleason Theorem). *Let G be a compact group acting on a normal space X with a closed invariant set A. Let G also act linearly on $\mathbb{R}^n$. Then any equivariant map $\phi : A \to \mathbb{R}^n$ extends to an equivariant map $\phi : X \to \mathbb{R}^n$.*

An *orbit* of a point x of X is $G(x) = \{g(x)|g \in G\}$. Then we see that $G/G_x \to G(x)$ is one-to-one and onto continuous function. Therefore, the orbit type is given by the conjugacy class of G_x in G. The set of orbit types form a set partially ordered by the reversing the inclusion ordering of the conjugacy classes of subgroups of G.