Chapter 3

Geometry and discrete groups

In this section, we will introduce basic materials in the Lie group theory and geometry and discrete group actions on the geometric spaces.

Geometry will be introduced as in the Erlangen program of Klein. We discuss projective geometry in some depth. Hyperbolic geometry will be given an emphasis by detailed descriptions of models. Finally, we discuss the discrete group actions, the Poincaré polyhedron theorem and the crystallographic group theory.

We will not go into details as these are somewhat elementary topics. A good source of the classical geometry is carefully written down in the book [Berger (2009)]. The rest of material is heavily influenced by the books [Ratcliffe (2006); Thurston (1997)]; however, we sketch the material.

3.1 Geometries

We will now describe classical geometries from Lie group action perspectives, as expounded in the Erlangen program of Felix Klein submerging all classical geometries under the theory of Lie group actions: We think of an (G, X)-*geometry* as the invariant properties of a manifold X under a group G acting on it transitively and effectively. Formally, the (G, X)-geometry is simply the pair (G, X) and we should know everything about the (G, X)-geometry from this pair.

Of course, there are many particular hidden treasures under this pair which should surface when we try to study them.

3.1.1 *Euclidean geometry*

The Euclidean space is $\mathbb{R}^n$ (or denoted $\mathbb{E}^n$) and the group $\mathbf{Isom}(\mathbb{R}^n)$ of rigid motions is generated by $\mathbb{O}(n)$ and T_n the translation group. In fact, we have an inner-product giving us a metric.

A system of linear equations gives us a subspace (affine or linear). Hence, we have a notion of points, lines, planes, and angles. Notice that these notions are invariantly defined under the group of rigid motions. These give us the set theoretical model